The Binomial and Black-Scholes Option Pricing Models: A Pedagogical Review with Vba Implementation

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Abstract—In this paper, a pedagogical review of two option pricing models is presented; specifically, the Binomial and the Black-Scholes pricing models. Theoretically these models converge for a very large number of exercise periods within a single option contract by virtue of the central limit theorem being based on the random walk and the Brownian motion processes respectively. This relationship is graphically illustrated by the use of an MS VBA implementation of the models.

Keywords—Binomial distribution, option pricing, Black-Scholes model, convergence, VBA.

1. Introduction

In recent years, the financial markets have improved considerably. Thus people can invest using various strategies or instruments to either reduce the risk of trading and investment and also maximize profit.

Many strategies have found their way into the financial market allowing individuals, corporate institutions and governments to reduce their risk and maximize profit on investments. Derivative instruments are mostly used in recent times.

Derivative instruments are agreements (contracts or financial instruments) that have their value determined by the price of another asset. The basic types of derivative instruments are the Forwards, Futures, Options and Swaps. They are traded on either Exchanges or on Over-the-counter (OTC) markets [1].

Among the various derivative instruments mentioned above, option trading gives the holder the right and not the obligation to buy or sell an underlying asset at a predetermined price at a specified future time. Options are mostly used to either reduce the risk of loss on a particular investment (hedging) or to maximize profit on an investment. Option trading gives an investor the ability to combine various strategies (in addition to the long and short positions) offered by all the types of derivatives. This underlines the flexibility of option trading.

Stock options are traded on most exchanges around the world. The value of stock options is dependent on the movement of stock prices. Thus an investor will take a position depending on whether prices of stocks move up or down. Investors will only exercise their option if they have a positive payoff. Stock traded options have fixed settlement terms, and a fixed time to exercise right.

Everything in the contract is negotiable: the quantity, type, delivery or settlement procedure of the underlying asset, the expiry date, and the strike price for the contract [2].

In this paper we will concentrate on option contracts where the holder can exercise his right on a specified expiration time.

1.2 Types of Option Contracts

The two major option contracts are the call option which gives the holder the right to buy, and the put option which gives the holder the right to sell [3].

The buyer of a call option pays a premium to the seller and, in return, has the right (but is not obligated) to buy a specific number and type of stock at a fixed price, before or at a given date.

The buyer of a put option pays a premium to the seller and, in return, has the right (but is not obligated) to sell a specific amount and type of oil at a fixed price, before or at a given date.

The fixed pre-determined price at which the holder of the option can buy or sell the underlying asset is usually known as the strike price or exercise price. The date agreed in the contracts is usually known as the expiration date, exercise date or the maturity of the option.
1.3 Option Trading Positions

There are four main option trading positions which give the right holder certain payoffs. These are defined below:

1.3.1 Long Call

A trader who believes that a stock's price will increase might buy the right to purchase a call option. This is to give him the right to buy the stock at lower price in the future but not necessarily buying the stock today. He would have no obligation to buy the stock. If the stock price at expiration is above the exercise price by more than the premium paid, he will profit. If the stock price at expiration is lower than the exercise price, he will let the call contract expire worthless, and only lose the amount of the premium. Thus the Payoff on a purchased call is given by \( \text{Max}(0, S_T - K) \). Thus the profit of the right holder is given by \( \text{Max}(0, S_T - K) - FV(\text{premium}) \). This is shown in figure 1:

![Figure 1. Pay-off and Profit (Long Call)](image)

1.3.2 Long Put

A trader who believes that a stock's price will decrease can buy the stock or instead sell a put. The trader selling a put has an obligation to buy the stock from the put buyer at the put buyer's option. If the stock price at expiration is above the exercise price, the short put position will make a profit in the amount of the premium. If the stock price at expiration is below the exercise price by more than the amount of the premium, the trader will lose money, with the potential loss being up to the full value of the stock. The payoff for this position is given by \( -\text{max}[0, K - S_T] \) and has a profit of \( FV(\text{premium}) - \text{max}[0, K - S_T] \). This is shown in Figure 4.

![Figure 4. Pay-off and Profit (Short put)](image)

1.3.3 Short Call

A trader who believes that a stock price will decrease, can sell the stock short or instead sell, or "write," a call. The trader selling a call has an obligation to sell the stock to the call buyer at the buyer's option. If the stock price decreases, the short call position will make a profit in the amount of the premium. If the stock price increases over the exercise price by more than the amount of the premium, the short will lose money, with the potential loss loss unlimited. The payoff of a written call is given by \(-\text{max}(0, S_T - K)\). Thus the profit is given by \(FV(\text{Premium}) - \text{max}(0, S_T - K)\). This is illustrated in Figure 3.

![Figure 3. Pay-off and Profit (Short call)](image)

1.3.4 Short Put

A trader who believes that a stock price will increase can buy the stock or instead sell a put. The trader selling a put has an obligation to buy the stock from the put buyer at the put buyer's option. If the stock price at expiration is above the exercise price, the short put position will make a profit in the amount of the premium. If the stock price at expiration is below the exercise price by more than the amount of the premium, the trader will lose money, with the potential loss being up to the full value of the stock.

The payoff for this position is given by \(-\text{max}[0, K - S_T]\) and has a profit of \(FV(\text{premium}) - \text{max}[0, K - S_T]\).

This is shown in Figure 4.

1.3.5 Option Pricing Parameters

There exist six factors that affect the price of a stock option namely:
1. The current stock price $S_0$ which is the prevailing market price of stock at expiration date.
2. The strike price $K$: which is the predetermined price at which the holder will exercise right
3. The time to expiration $T$: which is the time duration the holder has to exercise right
4. The volatility of the stock price $\sigma$: which measures the uncertainty of movement in the market.
5. The risk-free interest rate, $r$: which is the rate of investment on stock.
6. The dividends expected during the life of the option.

Variation in the above parameters affects the price of the option. The impacts on price of the option as a result of changes in the parameters are discussed below.

(a) **Stock Price**

If stock prices increase the value of a call option increases but the value for put options decreases. Thus, if stock prices decrease the value of a call option decreases but the value for put options increases. This is shown in Figure 5.

![Figure 5. Variation in Stock Price](image)

(b) **Time to Expiration**

As the time to expiration changes both call option and put option experience increase in value. Nevertheless the value of a call option becomes always greater than the put option. This is shown in Figure 6.

![Figure 6. Variation in Time to Expiration Price](image)

(c) **Volatility**

Volatility creates parallel increase between call and put options. This is shown in Figure 7.

![Figure 7. Variation in Volatility](image)

(d) **Risk Free Interest Rate**

The risk-free interest rate affects the price of an option in a less clear-cut way. As interest rates in the economy increase, the expected return required by investors from the stock tends also to increase. In addition, the present value of any future cash flow received by the holder of the option decreases. The combined impact of these two effects is to increase the value of call options and decrease the value of put options.

This is shown in Figure 8.

![Figure 8. Variation in Interest Rate](image)

2. **Models**

2.2 **The Black-Scholes Model**

Considering earlier difficulties in calculating option prices, Fischer Black, Myron Scholes and Robert Merton derived the Black-Scholes Model. This model has been the breakthrough in the option market and is widely used today.

2.2.1 **Black-Scholes Model Assumptions**

There are several assumptions underlying the Black-Scholes model of calculating options pricing. These assumptions are combined with the principle
that options pricing should provide no immediate gain to either seller or buyer [1].

The assumptions of the Black-Scholes Model are:

1. Stock pays no dividends
2. Options can only be exercised upon expiration
3. Market direction cannot be predicted, hence "Random Walk"
4. No commissions are charged in the transaction
5. Interest rates remain constant
6. Stock returns are normally distributed, thus volatility is constant over time.

2.2.2 Black-Scholes Pricing Formula

In financial terms, the price of an option is simply the present value of the future income stream that can be expected from holding the option contract [4]-[6].

The Black-Scholes formulas for the prices at time 0 of a European call option on a non-dividend paying stock and a European put option on a non-dividend paying stock are

\[ c = S_0 N(d_1) - Ke^{-rT} N(d_2) \]
\[ p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \]

Where

\[ d_1 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \]
\[ d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} - \sigma \sqrt{T} \]

The function \( N(x) \) is the cumulative probability distribution function for a standardized normal distribution. It is the probability that a variable with a standard normal distribution will be less than \( x \).

2.3 The Binomial Model

Given that stock prices’ moving up or down is denoted by \( u \) or \( d \) respectively, the binomial tree illustrates a one-step price movement in stock and option.

The notations used are explained as follows:

\( S_0 \): current stock price
\( S_0u \): new price of stock as a result of upward movement from \( S_0 \)
\( S_0d \): new price of stock as a result of downward movement from \( S_0 \)
\( f_u \): One step payoff from option if prices move up to \( S_0u \).
\( f_d \): One step payoff from option if prices move down to \( S_0d \).

Delta (\( \Delta \)): the number of shares needed to create a riskless portfolio.

The formulae for calculating the price of option using the binomial tree model is based on the riskless portfolio approach at a risk free interest rate. The riskless portfolio can be generated from the fact that there are only two securities (the stock and the stock option) with only two possible outcomes.

In creating a riskless portfolio for both long and short positions, \( \Delta \) (denoting the number of stocks needed to create a riskless portfolio) should be determined. The value of the portfolio at the end of the option life is \( S_0u\Delta - f_u \) for an upward movement and \( S_0d\Delta - f_d \) for downward movement.

For a riskless portfolio:

\[ S_0u\Delta - f_u = S_0d\Delta - f_d \]

With a risk free interest rate of \( r \), the present value of the portfolio is

\[ (S_0u\Delta - f_u)e^{-rT} \]

The payoff of the option is given as

\[ f = e^{-rT} \{ pf_u + (1 - p)f_d \} \]

Where, \( p \) is the probability that stock prices go up by \( u \) or go down by \( d \). This can be calculated as

\[ p = \frac{e^{rT} - d}{u - d} \]

Given that upward and downward movements (\( u \) and \( d \)) are by definite amounts.

The model can be generalized as

\[ f = e^{-rT} \{ p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd} \} \]

for a two-step binomial

2.3.1 Generalizing the Formula

The value of a stock at each \((i,j)\)th node is given as;
The value of the option is given as:

\[ f_{i,j} = e^{-r\Delta t} \left[ f_{i+1,j+1} + (1 - p) f_{i+1,j} \right] \]

Where \( f_{i,j} \) is the value of the option at the \((i,j)\)th node

**Call Payoff** = \( \text{Max}[0, K - S_T] \)

**Put Payoff** = \( \text{Max}[0, S_T - K] \)

3. Results

3.2 Black-Scholes Model Implementation

An Excel VBA implementation ([7] - [8]) of the formulae for the computation of option values using either the Black Scholes or the Binomial Models was carried out. The results are provided below in terms of Input and Output Forms in a GUI context of VB.

Thus in the computation of the value of an option using the Black-Scholes model the required parameter values namely, the Strike Price, Stock Price, Interest Rate, Volatility as well as the type of option being considered (Put or Call) must be entered as data through an input window as shown in Figure 9.

![Figure 9. Input Form for Black Scholes Model](image)

The corresponding output form in respect of the Black Scholes Model then displays the option value as shown in Figure 10.

![Figure 10. Output Form for Black Scholes Model](image)

3.3 Binomial Model Implementation

Again, in the computation of the value of an option using the Binomial Model the required parameter values namely, the Strike Price, Stock Price, Interest Rate, Volatility, Number of Periods as well as the type of option being considered (Put or Call and whether American or European style) must be entered as data through an input window as shown in Figure 11.

![Figure 11. Input Form for Binomial Model](image)

Clicking the “OK” button returns the value of the option in an output form shown Figure 12.
Figure 12 Output Form for Binomial Model

Its associated binomial tree showing the expected stock prices and option values at the different nodes is also displayed after the clicking the “Display Tree” button. Also for American options the program shows when exercising the option is profitable by returning the value of the option in red at that node. The generalized formulae for the model stated above were used for developing the following program formulas are employed:

In the example given, the specified number of periods was five (5). The program generated the option price at each node and the stock price at each node. At each node the price of the stock at that node is above the price of the option. This is shown in Figure 13.

Figure 13 Binomial Tree Output

The tree represents an American put. In the tree, the price of option which appears red indicates that the holder will benefit if he exercise option right. The tree can also be displayed for American call, European call and European put.

4. Discussion

4.2 Convergence of the Binomial to the Black-Scholes Model

As the number of exercise period between option contacts increases the binomial model converges to the Black Scholes model. It is based on the stochastic principle of random walk and Brownian motion. Though theoretically proven, using excel and VBA give more understanding to this proof. The diagram in Figure 14 displays the convergence of the Binomial to the Black-Scholes based on the inputs made in the program.

Figure 14 Convergence of Binomial to Black Scholes Model

5. Conclusion

In conclusion, the program can be used to value European call and put options, American call and put options using either the Binomial or Black-Scholes models. The application works well in producing the value of a non-dividend paying stock for the two types of options. The application also is very simple and easy to use. The graphical proof of convergence gives more understanding to the theoretical proof of convergence. Developing application such as this make it easier for everyone whether student of professional to use since it reduces time spent and errors attendant to a manual calculation.

The graphical output in relation to the binomial tree indicates optimal points where exercise of option right becomes profitable.

References


