Solving Time-Fractional Reaction–Diffusion Equation by Reduced Differential Transform Method

Aydin SECER

Mathematical Engineering, Yildiz Technical University
34210-Davutpaşa-Istanbul, Turkey
asecer@yildiz.edu.tr

Abstract—In this article, a novel numerical method is proposed for nonlinear partial differential equations with time fractional derivatives. The aim of the presented paper is to investigate the application of reduced differential transform method in order to solve the fractional reaction–diffusion equation with ecological parameters. This method needs less work in comparison with the traditional methods and decreases considerable volume of calculation. Comparing the methodology with some other known techniques shows that the present approach is effective and powerful.

Keywords— Fractional differential equation, Reduced differential transform method, Fractional reaction–diffusion equation.

1. Introduction

There is a long-standing interest in extending the classical calculus to non–integer orders [1-3]. Because fractional differential equations are suitable models for many physical problems. The development of this generalized calculus, however, has been consistently hampered because many fractional differential equations are nonlinear and have no exact analytical solutions that can be likened to the classical case. Many numerical schemes have been proposed over the years to approximate the solutions of fractional equations, for example, the Adomian decomposition method (ADM) [4], the variation iteration method (VIM) [5], the homotopy perturbation method (HPM) [6], the differential transform method (DTM) [7]. In this paper, we solve fractional reaction–diffusion equations by the reduced differential transform method [8, 9] which is presented to overcome the demerit of complex calculation of differential transform method (DTM). The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms.

Consider the nonlinear initial-boundary value time-fractional reaction–diffusion parabolic problems

\[ D^\mu U = DU_{ss} + m(1-U) \]  

subject to initial conditions

\[ u(x,0) = f(x) \]

where \( u \) is a function of \( x \) and \( t \), \( f(x) \) is a known analytic function. This equation was first introduced by Fisher as a model for the propagation of a mutant gene. It has wide application in the fields of logistic population growth, flame propagation, euro physiology, autocatalytic chemical reactions, branching Brownian motion processes, and nuclear reactor theory [10, 11, 12, and 13].

In this paper, we introduce reduced differential transform method to solve fractional reaction–diffusion equations.

We organized this paper as the following.

- In the next section, we begin by introducing the definition and the basic mathematical operations.
- In section 3, we give the definition and the properties of reduced differential transform method.
- We show how to apply the proposed method to a given fractional reaction–diffusion equations.
- In section 4, we apply the reduced differential transform method to solve test example in order to show its ability and efficiency.
- Finally we conclude this paper with a brief discussion in Section 5.

2. Basic Definitions

In this section, we present some basic definitions and properties of the fractional calculus [2, 3].

Definition 2.1. A real function \( f(x), x > 0 \), is said to be in the space \( C_\mu \), \( \mu \in R \) if there exists a real number \( p > \mu \), such that \( f(x) = x^pf_1(x) \), where \( f_1(x) \in C[0, \infty) \), and it said to be in the space

\[ C_\mu^m \iff f^m \in C_\mu, m \in N. \]  

(2.1)
**Definition 2.2.** The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f^m \in C_{\mu}, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\gamma + 1)} \int_0^x (x-t)^{\gamma-1} f(t)dt, \gamma > 0,$$

$$J^\alpha f(x) = f(x).$$

(2.2)

It has the following properties:

For $f^m \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > 1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$
3. $J^\alpha x = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} x^{\alpha+\gamma}$.

The Riemann–Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for the physical problems of the real world since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition which has the advantage of defining integer order initial conditions for fractional order differential equations.

**Definition 2.3.** The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt$$

(2.3)

for $m-1 < \alpha < m, m \in N, x > 0, f \in C^m_{\mu}$.

**Lemma 2.1.** If $m-1 < \alpha < m, m \in N$ and $f \in C^m_{\mu}, \mu \geq -1$, then

$$D^\alpha J^\alpha f(x) = f(x),$$

$$J^\alpha D^\alpha f(x) = f(x) = \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, x > 0.$$

(2.4)

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

**Definition 2.4.** For $m$ to be the smallest integer that exceeds $\alpha$, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = $$

$$\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^\alpha u(x,\xi)}{\partial x^\alpha} d\xi, m-1 < \alpha$$

(2.5)

and the space-fractional derivative operator of order $\beta > 0$ is defined as

$$D^\beta u(x,t) = \frac{\partial^\beta u(x,t)}{\partial x^\beta} = $$

$$\frac{1}{\Gamma(m-\beta)} \int_0^t (t-\Theta)^{m-\beta-1} \frac{\partial^\beta u(x,\Theta)}{\partial x^\beta} d\Theta, m-1 < \beta < m$$

(2.6)

**3. Reduced Differential Transform Method**

The reduced differential transform method (RDTM) [9] is applicable to a large class of nonlinear problems with approximations that converge rapidly to the actual solutions. A brief overview of this method is given below:

**Definition 3.1.** Let $u(x,t)$ be an analytic function that continuously differentiable with respect to $t$ and space $x$ in domain of interest. Define

$$U_j(x) = \frac{1}{\Gamma(k\alpha + 1)} \left[ \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=0}$$

(3.1)

where $\alpha$ is a parameter describing the order of time-fractional derivative in the Caputo sense and $t$-dimensional spectrum function $U_j(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_j(x)t^{k\alpha}$$

(3.2)

Combining equations (3.1) and (3.2), we have

$$u(x,t) = \sum_{k=1}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[ \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=0} \cdot t^{k\alpha}.$$  (3.3)

**Table 1 Reduced differential transformations**

<table>
<thead>
<tr>
<th>Functional Form</th>
<th>Transformed Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(x,t)$</td>
<td>$U_j(x) = \frac{1}{\Gamma(k\alpha + 1)} \left[ \frac{\partial^{k\alpha}}{\partial x^{k\alpha}} u(x,t) \right]_{t=0}$</td>
</tr>
</tbody>
</table>
To illustrate the RDTM, consider equation (1.1). According to Table 1, we can construct the following iteration formula:

$$\frac{\Gamma(k\alpha + \alpha) + 1}{\Gamma(k\alpha + 1)} U_{i+1}(x) = G_i(x) - R(U_i(x)) - N(U_i(x))$$  \hspace{1cm} (3.4)

where $R(U_i(x))$, $N(U_i(x))$ and $G_i(x)$ are the transformations of the functions $R(u(x,t))$, $R(u(x,t))$ and $g(x,t)$ respectively. From the initial condition (1.2), we have

$$U_0(x) = f(x)$$  \hspace{1cm} (3.5)

which can be substituted into (3.4) to calculate the $U_k(x)$ values whose inverse transformations $\{U_k(x)\}_{k=0}^{\infty}$ provide approximation of the solutions

$$\tilde{u}_k(x,t) = \sum_{i=0}^{N} U_i(x)t^i$$  \hspace{1cm} (3.6)

order of $m$. The exact solution of the problem is therefore given by

$$u(x,t) = \lim_{n \to \infty} \tilde{u}_n(x,t), i = 1, ..., n.$$  \hspace{1cm} (3.7)

### 4. Application

Consider the nonlinear time-fractional Fisher’s equation

$$D_t^\alpha u(x,t) = u(x,t) + 6u(x,t)(1-u(x,t))$$  \hspace{1cm} (4.1)

with initial condition

$$u(x,0) = \frac{1}{\left(1 + e^{x}\right)^2}$$  \hspace{1cm} (4.2)

where $u = u(x,t)$ is a function of the variables $x$ and $t$. Then, by using the basic properties of the reduced differential transformation, we can find the transformed form of equation (4.1) as

$$\frac{\Gamma(k\alpha + \alpha) + 1}{\Gamma(k\alpha + 1)} U_{k+1}(x) = \frac{\partial^2}{\partial x^2} U_k(x) + N_k(x)$$  \hspace{1cm} (4.3)

where $N_k(x)$ is transformed form of $6u(x,t)(1-u(x,t))$.

Using the initial condition (4.2), we have

$$U_0(x) = \frac{1}{\left(1 + e^{x}\right)^2}$$  \hspace{1cm} (4.4)

Now, substituting (4.4) into (4.3), we obtain the following $U_k(x)$ values successively

$$U_1(x) = \frac{10e^x}{\left(1 + e^{x}\right)^2} \frac{1}{\Gamma(\alpha + 1)}$$  

$$U_2(x) = \frac{50e^x(2e^x - 1)}{\left(1 + e^{x}\right)^2} \frac{1}{\Gamma(2\alpha + 1)}$$  

$$U_3(x) = \frac{750e^x(4e^{2x} - 7e^x + 1)}{3(1 + e^{x})^4} \frac{1}{\Gamma(3\alpha + 1)}$$  

$$U_4(x) = \frac{15000e^x(8e^{3x} - 33e^{2x} + 18e^x - 1)}{12(1 + e^{x})^6} \frac{1}{\Gamma(4\alpha + 1)}$$  

$$U_k(x) = \frac{1}{k!} \left[ \frac{\partial^k}{\partial x^k} \left( \frac{1}{\left(1 + e^{x}\right)^2}\right) \frac{1}{\Gamma(k\alpha + 1)} \right]$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k t^k$$
\[
\sum_{k=0}^{\infty} \frac{\partial^k}{\partial t^k} \left( \frac{1}{(1+e^{-st})^2} \right) \frac{e^{\alpha k}}{\Gamma(\alpha k + 1)}.
\]

From (4.5) we obtain the exact solution \( u(x,t) = \frac{1}{(1+e^{-st})^2} \) as shown in Figure 1.

The approximate solutions for Eq. (4.1) obtained for different values of \( \alpha \) using the RDTM. The values of \( \alpha = 1 \) is the only case for the known exact solution.

**Figure 1** - The exact solution of nonlinear time-fractional Fisher’s equation for \( \alpha = 1 \).

**5. Conclusion**

This article is primarily concerned with the construction of approximate analytical solutions for fractional reaction–diffusion equation by using the reduced differential transform method. One example shows that the reduced differentials transform method is a powerful mathematical tool to solve time fractional reaction–diffusion equation. It is also a promising method to overcome the solution for other broad class of nonlinear equations. This method often produces rapidly converging sequences that converge to the exact solutions, thus making it a significant improvement over existing methods in solving fractional partial differential equations.

**References**


