Theory of Fuzzy Soft Sets from a New Perspective
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Abstract -- The theory of fuzzy sets introduced by Zadeh should actually have been a generalization of the classical theory of sets in the sense that the theory of sets should have been a special case of the theory of fuzzy sets. Unfortunately, this is not the case. It has been accepted that for a fuzzy set A and its complement Aᶜ, neither A∩Aᶜ is the null set, nor A∪Aᶜ is the universal set. Whereas the operations of union and intersection of two crisp sets are indeed special cases of the corresponding operations of two fuzzy sets, they end up giving peculiar results while defining A∩Aᶜ and A∪Aᶜ. In this regard, H. K. Baruah proposed that in the current definition of the complement of a fuzzy set, fuzzy membership function and fuzzy membership value had been taken to be the same, which led to the conclusion that the fuzzy sets do not follow the set theoretic axioms of exclusion and contradiction. For the complement of a normal fuzzy set, fuzzy membership function and fuzzy membership value are two different things, and the complement of a normal fuzzy set has to be defined accordingly. H. K. Baruah has reintroduced the definition of fuzzy set and redefined the complement of a fuzzy set accordingly. In 1999, Molodstov introduced the theory of soft sets, which can be seen as a new mathematical approach to vagueness. The theory of fuzzy soft set, a more generalized concept, initiated by Maji, is a combination of fuzzy set and soft set. In this paper, we improve the notion of union and intersection of fuzzy sets proposed by Baruah and generalize the concept of complement of a fuzzy set when the fuzzy reference function is not zero. We are defining arbitrary fuzzy union and intersection using the definition of fuzzy sets given by Baruah and then we are using this definition of fuzzy set and newly defined complement in the fuzzy sets occurring in a fuzzy soft set so that fuzzy soft sets, too, follow the axioms of exclusion and contradiction. Accordingly we propose some new notions regarding fuzzy subset, null fuzzy soft set, absolute fuzzy soft set, complement of a fuzzy soft set. We put forward definitions of “AND” and “OR” operations for an arbitrary collection of fuzzy soft sets and finally DeMorgan inclusions and DeMorgan laws proved by Ahmad and Kharal in fuzzy soft set theory have been verified according to our new notions.

Key words: Fuzzy set, fuzzy soft set, null fuzzy soft set, absolute fuzzy soft set, complement of a fuzzy soft set.

1. Introduction

Fuzzy Set Theory was introduced by Lofti Zadeh [5] in 1965 and it was specifically designed to mathematically represent uncertainty and vagueness with formalized logical tools for dealing with the imprecision inherent in many real world problems. Fuzzy sets are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of a degree.

In 1999, Molodstov [2] initiated a novel concept known as Soft Set as a new mathematical tool for dealing with uncertainties. He pointed out that the important existing theories viz. Probability Theory, Fuzzy Set Theory, Intuitionistic Fuzzy Sets, Rough Set Theory etc which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. These theories cannot be successfully used to solve complicated problems in the fields of engineering, social science, economics, medical science etc. He further pointed out that the reason for these difficulties is, possibly, the inadequacy of the parameterization tool of the theory. The Soft Set Theory introduced by Molodstov is free of the difficulties present in these theories. The absence of any restrictions on the approximate description in Soft Set Theory makes this theory very convenient and easily applicable. Zadeh’s fuzzy sets may be considered as a special case of the soft sets.

In recent years the researchers have contributed a lot towards the fuzzification of soft set theory. Maji et al. [7] proposed the concept of fuzzy soft sets and developed some properties of fuzzy soft sets. They introduced some properties regarding fuzzy soft union, intersection, complement of a fuzzy soft set, De Morgan Law etc. These results were further revised and improved by Ahmad and Kharal [1]. They defined arbitrary fuzzy soft union and intersection and proved De Morgan Inclusions and De Morgan Laws in Fuzzy Soft Set Theory. The theory of fuzzy sets initiated by Zadeh [5] should actually have been a generalization of the classical theory of sets in the sense that the theory of sets should have been a special case of the theory of fuzzy sets. Unfortunately, this is not the case. It has been accepted that for a fuzzy set A and its
complement $A^c$, neither $A \cap A^c$ is the null set, nor $A \cup A^c$ is the universal set. Whereas the operations of union and intersection of two crisp sets are indeed special cases of the corresponding operations of two fuzzy sets, they end up giving peculiar results while defining $A \cap A^c$ and $A \cup A^c$. In this regard H. K. Baruah [4] has reintroduced the definition of fuzzy sets which enables us to define complement of fuzzy set in a way that gives us $A \cap A^c$ the fuzzy null set and $A \cup A^c$ the universal set. We agree with him because the Zadehian definition fails to describe the complement of a fuzzy set correctly and hence all sorts of mathematical and applicational matters in which this definition had been used must also be wrong. To continue applying a wrong axiom just because it has been arrogantly believed to true for nearly half a century goes totally against the philosophy of mathematics. Since fuzzy soft set is generalization of soft set and fuzzy set, the concept of complement of a fuzzy soft set and all mathematical and applicational matters in which this definition had been used have to be reintroduced so as to avoid those complications faced only because of a wrong axiom that we believe to be true. Accordingly we put forward a new definition of complement of a fuzzy soft set in a way that gives $(F, A) \cap (F, A)^c = \emptyset$, the null fuzzy soft set, and $(F, A) \cup (F, A)^c = A$, the absolute fuzzy soft set and thereby try to reintroduce the matters where the definition of complement of a fuzzy soft set had been used.

2. Preliminaries

H. K. Baruah [4] reintroduced the definition of fuzzy set in the following manner - According to him, to define a fuzzy set, two functions namely fuzzy membership function and fuzzy reference function are necessary. Fuzzy membership value is the difference between fuzzy membership function and reference function. Fuzzy membership function and fuzzy membership value are two different things. In the Zadehian definition of complementation, these two things have been taken to be the same, and that is where the error lies.

Let $\mu_1(x)$ and $\mu_2(x)$ be two functions, $0 \leq \mu_2(x) \leq \mu_1(x) \leq 1$. For a fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x); x \in U\}$, we would call $\mu_1(x)$ the fuzzy membership function, and $\mu_2(x)$ a reference function, such that $\{\mu_1(x) - \mu_2(x)\}$ is the fuzzy membership value for any $x$. In the definition of complement of a fuzzy set, the fuzzy membership value and the fuzzy membership function have to be different, in the sense that for a usual fuzzy set the membership value and the membership function are of course equivalent.

Let $A(\mu_1, \mu_2) = \{x, \mu_1(x), \mu_2(x); x \in U\}$ and $B(\mu_3, \mu_4) = \{x, \mu_3(x), \mu_4(x); x \in U\}$ be two fuzzy sets defined over the same universe $U$. Then the operations intersection and union are defined as

$$A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4) = \left\{ x, \min(\mu_1(x), \mu_3(x)), \max(\mu_2(x), \mu_4(x)); x \in U \right\}$$

and

$$A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4) = \left\{ x, \max(\mu_1(x), \mu_3(x)), \min(\mu_2(x), \mu_4(x)); x \in U \right\}$$

Two fuzzy sets $C = \{x, \mu_C(x); x \in U\}$ and $D = \{x, \mu_D(x); x \in U\}$ in the usual definition would be expressed as

$$C(\mu_C, 0) = \{x, \mu_C(x), 0; x \in U\}$$

and

$$D(\mu_D, 0) = \{x, \mu_D(x), 0; x \in U\}$$

Accordingly, we have

$$C(\mu_C, 0) \cap D(\mu_D, 0) = \left\{ x, \min(\mu_C(x), \mu_D(x)), \max(0, 0); x \in U \right\}$$

which in the usual definition is nothing but $C \cap D$.

Similarly, we have

$$C(\mu_C, 0) \cup D(\mu_D, 0) = \left\{ x, \max(\mu_C(x), \mu_D(x)), \min(0, 0); x \in U \right\}$$

which in the usual definition is nothing but $C \cup D$. Thus we have seen that for union and intersection of two fuzzy sets, the extended definition leads to the union and intersection under the standard definition.
These new definitions lead to the conclusion that for usual fuzzy sets

\[
A(\mu, 0) = \{x, \mu(x), 0; x \in U\}
\]

and

\[
B(1, \mu) = \{x, 1, \mu(x); x \in U\}
\]
defined over the same universe U we have

\[
A(\mu, 0) \cap B(1, \mu) = \{x, \min(\mu(x), 1), \max(0, \mu(x)); x \in U\}
\]

\[
= \{x, \mu(x), \mu(x); x \in U\}, \text{ which is nothing but the null set } \varnothing.
\]

and

\[
A(\mu, 0) \cup B(1, \mu) = \{x, \max(\mu(x), 1), \min(0, \mu(x)); x \in U\}
\]

\[
= \{x, 1, 0; x \in U\}, \text{ which is nothing but the universal set } U.
\]

This means if we define a fuzzy set \((A(\mu, 0))^c = \{x, 1, \mu(x); x \in U\}\), it is nothing but the complement of \(A(\mu, 0) = \{x, \mu(x), 0; x \in U\}\).

Thus it can be concluded that

\[
A(\mu, 0) \cap (A(\mu, 0))^c = \varnothing, \text{ the null set and}
\]

\[
A(\mu, 0) \cup (A(\mu, 0))^c = U, \text{ the universal set.}
\]

Baruah [3, 4] has thus reintroduced the Zadehian theory of fuzzy sets in a new perspective which would definitely bring about a revolution in the history of fuzzy set theory. However the definitions of union and intersection proposed by Baruah [3, 4] need some modifications so as to avoid degenerate cases. Let us take the following example.

**Example 1.**

Let \(U = \{a, b, c\}\) be the universal set. We take two fuzzy sets \(A\) and \(B\) as –

\[
A(\mu_1, \mu_2) = \{(a, 0.1, 0), (b, 0.2, 0.1), (c, 0.4, 0.2)\}
\]

and

\[
B(\mu_3, \mu_4) = \{(a, 0.9, 0.3), (b, 0.5, 0.3), (c, 0.5, 0.3)\}.
\]

Then

\[
(A \cap B)(\mu_5, \mu_6)
\]

\[
= \{x, \mu_5(x), \mu_6(x); x \in U\}
\]

\[
= \{(a, 0.1, 0.3), (b, 0.2, 0.3), (c, 0.4, 0.3)\}
\]

We have seen that \(\mu_5(a) < \mu_6(a)\), \(\mu_5(b) < \mu_6(b)\), i.e., fuzzy membership functions of \(a\) and \(b\) are less than their corresponding fuzzy reference functions, which is going against our assumption that \(\mu_5(x) \geq \mu_6(x) \forall x \in U\).

Again

\[
(A \cup B)(\mu_7, \mu_8)
\]

\[
= \{x, \mu_7(x), \mu_8(x); x \in U\}
\]

\[
= \{(a, 0.9, 0.0), (b, 0.5, 0.1), (c, 0.5, 0.2)\}, \text{ which is not true.}
\]

To avoid such degenerate cases, we propose the following modification in the definitions of union and intersection.

**Definition 1.**

Let

\[
A(\mu_1, \mu_2) = \{x, \mu_1(x), \mu_2(x); x \in U\}
\]

and

\[
B(\mu_3, \mu_4) = \{x, \mu_3(x), \mu_4(x); x \in U\}
\]

be two fuzzy sets defined over the same universe \(U\). To avoid degenerate cases we assume that

\[
\min(\mu_1(x), \mu_3(x)) \geq \max(\mu_2(x), \mu_4(x)) \forall x \in U.
\]

Then the operation intersection is defined as

\[
A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4) = \{x, \min(\mu_1(x), \mu_3(x)), \max(\mu_2(x), \mu_4(x)); x \in U\}
\]

If for some \(x \in U\),

\[
\min(\mu_1(x), \mu_3(x)) < \max(\mu_2(x), \mu_4(x)),
\]

then our conclusion is that \(A \cap B = \varnothing\).

If for some \(x \in U\),

\[
\min(\mu_1(x), \mu_3(x)) = \max(\mu_2(x), \mu_4(x)),
\]

then also \(A \cap B = \varnothing\).

Further, we define the operation union, with

\[
\min(\mu_1(x), \mu_3(x)) \geq \max(\mu_2(x), \mu_4(x)) \forall x \in U
\]

as

\[
A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4) = \{x, \max(\mu_1(x), \mu_3(x)), \min(\mu_2(x), \mu_4(x)); x \in U\}
\]

Also, our another conclusion is that if for some \(x \in U\), \(\min(\mu_1(x), \mu_3(x)) < \max(\mu_2(x), \mu_4(x))\) then the union of the fuzzy sets \(A\) and \(B\) cannot be expressed as one single fuzzy set. The union, however, can be expressed in one single fuzzy set if \(\min(\mu_1(x), \mu_3(x)) = \max(\mu_2(x), \mu_4(x))\).

Above example makes this clear. For usual fuzzy sets with reference function 0, it is quite obvious to see that the above conditions for defining union and intersection hold good.

Now we give the following definition of fuzzy subset which gives us the usual definition of fuzzy subset as a particular case.

**Definition 2.**

Let

\[
A(\mu_1, \mu_2) = \{x, \mu_1(x), \mu_2(x); x \in U\}
\]

and

\[
B(\mu_3, \mu_4) = \{x, \mu_3(x), \mu_4(x); x \in U\}
\]

be two fuzzy sets defined over the same universe \(U\). The fuzzy set \(A(\mu_1, \mu_2)\) is a subset of the fuzzy set
\[ B(\mu_3, \mu_4) \text{ if } \forall x \in U, \mu_1(x) \leq \mu_3(x) \text{ and } \mu_4(x) \leq \mu_2(x). \]

Two fuzzy sets \( C = \{x, \mu_C(x) : x \in U\} \) and \( D = \{x, \mu_D(x) : x \in U\} \) in the usual definition would be expressed as \( C(\mu_C, 0) = \{x, \mu_C(x), 0 : x \in U\} \) and \( D(\mu_D, 0) = \{x, \mu_D(x), 0 : x \in U\} \).

Accordingly, we have \( C(\mu_C, 0) \subseteq D(\mu_D, 0) \) if \( \forall x \in U, \mu_C(x) \leq \mu_D(x) \), which can be obtained by putting \( \mu_2(x) = \mu_4(x) = 0 \) in our new definition.

It can be verified that for fuzzy sets \( A(\mu_1, \mu_2), B(\mu_3, \mu_4), C(\mu_5, \mu_6) \) over the same universe \( U \), the following propositions are valid.

**Proposition 1.**

1. \( A(\mu_1, \mu_2) \subseteq B(\mu_1, \mu_4), B(\mu_3, \mu_4) \subseteq C(\mu_5, \mu_6) \Rightarrow A(\mu_1, \mu_2) \subseteq C(\mu_5, \mu_6) \)

2. \( A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4) \subseteq A(\mu_1, \mu_2), A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4) \subseteq B(\mu_3, \mu_4) \)

3. \( A(\mu_1, \mu_2) \subseteq A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4), B(\mu_3, \mu_4) \subseteq A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4) \)

4. \( A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4) = \mu_1(x), \mu_2(x) \Rightarrow A(\mu_1, \mu_2) \subseteq B(\mu_3, \mu_4) \)

5. \( A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4) = B(\mu_3, \mu_4) \)

Baruah [3, 4] has defined complement of a fuzzy set when the fuzzy reference function is always 0. We now generalize the concept of complement of a fuzzy set when the fuzzy reference function is not zero and it can be verified that our definition yields Baruah’s definition [3, 4] when we take fuzzy reference function \( = 0 \) \( \forall x \in U \).

**Definition 3.**

Let \( A(\mu_1, \mu_2) = \{x, \mu_1(x), \mu_2(x) : x \in U\} \) be a fuzzy set defined over the universe \( U \). Then the complement of the extended fuzzy set \( A(\mu_1, \mu_2) \) is defined as

\[
(A(\mu_1, \mu_2))^c = \{x, \mu_1(x), \mu_2(x) : x \in U\}^c = \{x, \mu_2(x), 0 : x \in U\} \cup \{x, 1, \mu_1(x) : x \in U\}
\]

Membership value of \( x \) in \( (A(\mu_1, \mu_2))^c \) is given by \( \mu_2(x) + (1 - \mu_1(x)) = 1 + \mu_2(x) - \mu_1(x) \)

If \( \mu_2(x) = 0 \), then membership value of \( x \) is \( 1 + 0 - \mu_1(x) = 1 - \mu_1(x) \)

For \( x \in U \), \( \min(\mu_2(x), 1) < \max(0, \mu_1(x)) \), so the union of these two fuzzy sets cannot be expressed as one single fuzzy set.

We can show that our definition of complement of a fuzzy set yields Baruah’s definition [3, 4] when we take fuzzy reference function \( = 0 \) \( \forall x \in U \). Further, it can be verified that in our definition the following properties are valid.

**1. Axioms of Contradiction and exclusion**

i. \( A(\mu_1, \mu_2) \cap (A(\mu_1, \mu_2))^c = \emptyset \) (Contradiction)

ii. \( A(\mu_1, \mu_2) \cup (A(\mu_1, \mu_2))^c = U \) (Exclusion)

**2. De Morgan Laws**

i. \( (A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4))^c = (A(\mu_1, \mu_2))^c \cap (B(\mu_3, \mu_4))^c \)

ii. \( (A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4))^c = (A(\mu_1, \mu_2))^c \cup (B(\mu_3, \mu_4))^c \)

**3. Involution**

\[
((A(\mu_1, \mu_2))^c)^c = A(\mu_1, \mu_2)
\]

**4. \( \phi^c = U \), \( U^c = \phi \)**

The same results for usual fuzzy sets with fuzzy reference function \( = 0 \) can be seen as particular cases of what we are giving.

We now proceed to define arbitrary fuzzy union and intersection using the definition of fuzzy sets given by Baruah [3, 4].

**Definition 4.**

Let \( \mathcal{F} = \{A_i(\mu_{i1}, \mu_{i2}) : i \in I\} \) be a family of fuzzy sets over the same universe \( U \). To avoid degenerate cases we assume that \( \min(\mu_{i1}(x)) \geq \max(\mu_{i2}(x)) \forall x \in U \). Then the union of fuzzy sets in \( \mathcal{F} \) is a fuzzy set given by

\[
\bigcup_i A_i(\mu_{i1}, \mu_{i2}) = \{x, \max(\mu_{i1}(x)), \min(\mu_{i2}(x)) : x \in U\}
\]
And the intersection of fuzzy sets in $\mathcal{Z}$ is a fuzzy set given by
$$\bigcap_i A_i(\mu_1, \mu_2) = \{x, \min(\mu_i(x)), \max(\mu_i(\mu_i(x)), x \in U\}$$

These definitions of arbitrary fuzzy union and fuzzy intersection lead to the following DeMorgan Laws.

**Proposition 2.**
Let $\mathcal{Z} = \{A_i(\mu_1, \mu_2) | i \in I\}$ be a family of fuzzy sets over the same universe $U$. To avoid degenerate cases we assume that $\min(\mu_1(x)) \geq \max(\mu_2(x)) \forall x \in U$. Then
$$1. \bigcup_i A_i(\mu_1, \mu_2)^c = \bigcap_i A_i(\mu_1, \mu_2)^c$$
$$2. \bigcap_i A_i(\mu_1, \mu_2)^c = \bigcup_i A_i(\mu_1, \mu_2)^c$$

Let us now have a new look at fuzzy soft set in the light of above definitions. Molodstov [2] defined soft set as follows.

**Definition 5 [2]**
A pair $(F, E)$ is called a soft set (over $U$) if and only if $F$ is a mapping of $E$ into the set of all subsets of the set $U$.

In other words, the soft set is a parameterized family of subsets of the set $U$. Every set $F(e), e \in E$, from this family may be considered as the set of $e$ - elements of the soft set $(F, E)$, or as the set of $e$ - approximate elements of the soft set.

Maji et al. [7] defined fuzzy soft set as follows.

**Definition 6 [7]**
A pair $(F, A)$ is called a fuzzy soft set over $U$ where $F : A \rightarrow \mathcal{P}(U)$ is a mapping from $A$ into $\mathcal{P}(U)$.

We strongly believe that in order to represent the fuzzy sets present in a fuzzy soft set, the definition of fuzzy set proposed by Baruah [3, 4] should be used and as such throughout our discussion we are using the same. For a usual fuzzy set fuzzy membership value and the fuzzy membership function are not different because the value of the function is counted from zero in the usual case. We give one example below based on this new notion.

**Example 2.**
Let $U = \{c_1, c_2, c_3, c_4\}$ be the set of four cars under consideration and
$$E = \{e_1(\text{costly}), e_2(\text{Beautiful}), e_3(\text{Fuel Efficient}), e_4(\text{Modern Technology}), e_5(\text{Luxurious})\}$$
be the set of parameters and $A = \{e_1, e_2, e_3\} \subseteq E$. Then
$$F(e_1) = \left\{\left(c_1, 0.7, 0\right), \left(c_2, 0.1, 0\right)\right\},$$
$$F(e_2) = \left\{\left(c_1, 0.8, 0\right), \left(c_2, 0.6, 0\right), \left(c_3, 0.1, 0\right), \left(c_4, 0.5, 0\right)\right\},$$
$$F(e_3) = \left\{\left(c_1, 0.1, 0\right), \left(c_2, 0.2, 0\right), \left(c_3, 0.7, 0\right), \left(c_4, 0.3, 0\right)\right\}\right.$$ is the fuzzy soft set representing the ‘attractiveness of the car’ which Mr. X is going to buy.

Ahmad and Kharal [1] defined fuzzy soft class as follows.

**Definition 7 [1]**
Let $U$ be a universe and $E$ a set of attributes. Then the pair $(U, E)$ denotes the collection of all fuzzy soft sets on $U$ with attributes from $E$ and is called a fuzzy soft class.

Maji et al. [7] gave the definition of null fuzzy soft set and absolute fuzzy soft set in the following way.

**Definition 8 [7]**
A soft set $(F, A)$ over $U$ is said to be null fuzzy soft set denoted by $\emptyset$ if $\forall e \in A, F(e)$ is the null fuzzy set $\emptyset$ of $U$ where $\emptyset(x) = 0 \forall x \in U$.

**Definition 9 [7]**
A soft set $(F, A)$ over $U$ is said to be absolute fuzzy soft set denoted by $\tilde{A}$ if $\forall e \in A, F(e)$ is the null fuzzy set $\tilde{A}$ of $U$ where $\tilde{A}(x) = 1 \forall x \in U$.

We have already come to the conclusion that the fuzzy membership value and fuzzy membership function for the complement of a fuzzy set are two different things although for a usual fuzzy set they are not different because the value of the function is counted from 0 in the
usual case. For a null fuzzy set, the fuzzy membership value i.e. the difference of fuzzy membership function and reference function has to be 0 and for absolute fuzzy set, the fuzzy membership value has to be land not the membership function as is done in the usual case. As such we have no other way than to redefine null fuzzy soft set and absolute fuzzy soft set based on this new notion.

From our stand point, we redefine Null Fuzzy Soft Set and Absolute Fuzzy Soft Set as given below.

**Definition 10.**

A fuzzy soft set \( (F, A) \) over \( U \) is said to be null fuzzy soft set (with respect to the parameter set \( A \)), denoted by \( \tilde{F} \) if \( \forall \varepsilon \in A, F(\varepsilon) = \) the null fuzzy set \( \varphi \).

**Definition 11.**

A fuzzy soft set \((F, A)\) over \(U\) is said to be absolute fuzzy soft set (with respect to the parameter set \(A\)), denoted by \(\sim A\) if \(\forall \varepsilon \in A, F(\varepsilon)\) is the absolute fuzzy set \(U\).

Maji et al. [7] defined union, intersection, “AND” and “OR” of fuzzy soft sets as follows.

**Definition 12 [7]**

Union of two fuzzy soft sets \((F, A)\) and \((G, B)\) in a soft class \((U, E)\) is a fuzzy soft set \((H, C)\) where \(C = A \cup B\) and \(\forall \varepsilon \in C\),

\[
H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } x \in A-B \\ G(\varepsilon), & \text{if } x \in B-A \\ F(\varepsilon) \cup G(\varepsilon), & \text{if } x \in A \cap B \end{cases}
\]

and is written as \((F, A) \cup (G, B) = (H, C)\).

**Definition 13 [7]**

Intersection of two fuzzy soft sets \((F, A)\) and \((G, B)\) in a soft class \((U, E)\) is a fuzzy soft set \((H, C)\) where \(C = A \cap B\) and \(\forall \varepsilon \in C\),

\[
H(\varepsilon) = F(\varepsilon) \cup G(\varepsilon) \quad \text{as both are same fuzzy set, and is written as } (F, A) \cap (G, B) = (H, C).
\]

Ahmad and Kharal [1] pointed out that generally \(F(\varepsilon)\) or \(G(\varepsilon)\) may not be identical. Moreover in order to avoid the degenerate case, he proposed that \(A \cap B\) must be non-empty and thus revised the above definition as follows.

**Definition 14 [1]**

Let \((F, A)\) and \((G, B)\) be two fuzzy soft sets in a soft class \((U, E)\) with \(A \cap B \neq \phi\). Then Intersection of two fuzzy soft sets \((F, A)\) and \((G, B)\) in a soft class \((U, E)\) is a fuzzy soft set \((H, C)\) where \(C = A \cap B\) and \(\forall \varepsilon \in C\),

\[
H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon) \quad \text{and we write } (F, A) \cap (G, B) = (H, C).
\]

**Definition 15 [7]**

If \((F, A)\) and \((G, B)\) be two fuzzy soft sets, then “(F, A) AND (G, B)” is a fuzzy soft set denoted by \((F, A) \cap (G, B)\) and is defined by \((F, A) \cap (G, B) = (H, A \times B)\),

\[
H(\alpha, \beta) = F(\alpha) \cap G(\beta) \quad \forall \alpha \in A \quad \text{and} \quad \forall \beta \in B
\]

where \(\cap\) is the operation fuzzy intersection of two fuzzy sets.

**Example 3.**

Let \(U = \{1, 2, 3, 4\}\) be the set of four cars under consideration and \(E = \{e_1, e_2, e_3\}\)

\[
E = \{e_1, e_2, e_3, e_4\}
\]

be the set of parameters and \(A = \{e_1, e_2, e_3\} \subseteq E\) and \(B = \{e_3, e_4\} \subseteq E\)

Then \((F, A)\)

\[
F(1) = \{(c_1, 0.7), (c_2, 0.1), (c_3, 0.2), (c_4, 0.6)\},
\]

\[
F(2) = \{(c_1, 0.8), (c_2, 0.6), (c_3, 0.1), (c_4, 0.5)\},
\]

\[
F(3) = \{(c_1, 1), (c_2, 0.2), (c_3, 0.7), (c_4, 0.3)\},
\]

and \((G, B)\)

\[
G(3) = \{(c_1, 0.5), (c_2, 0.1), (c_3, 0.4), (c_4, 0.0)\},
\]

\[
G(4) = \{(c_1, 0.4), (c_2, 0.2), (c_3, 0.9), (c_4, 0.2)\}
\]

\[
A \times B = \{(e_1, e_3), (e_1, e_4), (e_2, e_3), (e_2, e_4), (e_3, e_4), (e_4, e_4)\}
\]

\[
(F, A) \cap (G, B) = (H, A \times B),
\]

where \(H(\alpha, \beta) = F(\alpha) \cap G(\beta) \quad \forall \alpha \in A \quad \text{and} \quad \forall \beta \in B\).
\[ H(e_1, e_3) = F(e_1) \cap F(e_3), \quad H(e_1, e_4) = F(e_1) \cap F(e_4), \quad O(e_1, e_3) = F(e_1) \cup F(e_3), \quad O(e_1, e_4) = F(e_1) \cup F(e_4) \]

\[ H(e_2, e_3) = F(e_2) \cap F(e_3), \quad H(e_2, e_4) = F(e_2) \cap F(e_4), \quad O(e_2, e_3) = F(e_2) \cup F(e_3), \quad O(e_2, e_4) = F(e_2) \cup F(e_4) \]

\[ H(e_3, e_3) = F(e_3) \cap F(e_3), \quad H(e_3, e_4) = F(e_3) \cap F(e_4), \quad O(e_3, e_3) = F(e_3) \cup F(e_3), \quad O(e_3, e_4) = F(e_3) \cup F(e_4) \]

\[ H(e_1, e_4) = \{(c_1, 0.4, 0), (c_2, 0.1, 0), (c_3, 0.2, 0), (c_4, 0.2, 0)\}, \quad O(e_1, e_4) = \{(c_1, 0.8, 0), (c_2, 0.6, 0), (c_3, 0.9, 0), (c_4, 0.5, 0)\}, \]

\[ H(e_2, e_3) = \{(c_1, 0.5, 0), (c_2, 0.1, 0), (c_3, 0.1, 0), (c_4, 0.0)\}, \quad O(e_2, e_3) = \{(c_1, 0.5, 0), (c_2, 0.4, 0), (c_4, 0.0)\}, \]

\[ H(e_2, e_4) = \{(c_1, 0.4, 0), (c_2, 0.2, 0), (c_3, 0.1, 0), (c_4, 0.2, 0)\}, \quad O(e_2, e_4) = \{(c_1, 0.4, 0), (c_2, 0.2, 0), (c_3, 0.9, 0), (c_4, 0.3, 0)\} \]

\[ H(e_3, e_3) = \{(c_1, 0.5, 0), (c_2, 0.1, 0), (c_3, 0.4, 0), (c_4, 0.0)\}, \quad H(e_3, e_4) = \{(c_1, 0.1, 0), (c_2, 0.2, 0), (c_3, 0.7, 0), (c_4, 0.2, 0)\} \]

The following propositions on “AND” and “OR” Operations in Fuzzy soft sets according to our new notion are straight forward.

**Proposition 3.**

Let \((F_1, A_1), (F_2, A_2), (F_3, A_3)\) be three fuzzy soft sets in a fuzzy soft class \((U, E)\). Then the following properties hold good according to our new notion of extended fuzzy set.

(i) **Associative Property**

\[(a) \quad (F_1, A_1) \cap (F_2, A_2) \cap (F_3, A_3) = ((F_1, A_1) \cap (F_2, A_2)) \cap (F_3, A_3) \]

\[(b) \quad (F_1, A_1) \cup (F_2, A_2) \cup (F_3, A_3) = ((F_1, A_1) \cup (F_2, A_2)) \cup (F_3, A_3) \]

(ii) **Distributive Property**

\[(a) \quad (F_1, A_1) \cap (F_2, A_2) \cap (F_3, A_3) = ((F_1, A_1) \cap (F_2, A_2)) \cap (F_3, A_3) \]

\[(b) \quad (F_1, A_1) \cup (F_2, A_2) \cup (F_3, A_3) = ((F_1, A_1) \cup (F_2, A_2)) \cup (F_3, A_3) \]

Now we come to the definition of complement of a fuzzy soft set. Maji et al. [7] defined the complement of a fuzzy soft set in the following way.

**Definition 16** [7]

If \((F, A)\) and \((G, B)\) be two fuzzy soft sets, then \((F, A) \lor (G, B)\) is a fuzzy soft set denoted by \((F, A) \lor (G, B)\) and is defined by

\[(F, A) \lor (G, B) = (O, A \times B) , \quad \text{where } O(\alpha, \beta) = F(\alpha) \lor G(\beta), \forall \alpha \in A \quad \text{and} \quad \forall \beta \in B , \]

where \(\lor\) is the operation fuzzy union of two fuzzy sets.

**Example 4.**

In the above example,

\[(F, A) \lor (G, B) = (O, A \times B), \quad \text{where } O(\alpha, \beta) = F(\alpha) \lor G(\beta), \forall \alpha \in A \quad \text{and} \quad \forall \beta \in B \]

\[\{O(e_1, e_3) = F(e_1) \lor F(e_3), O(e_1, e_4) = F(e_1) \lor F(e_4)\}, \]

\[O(e_2, e_3) = F(e_2) \lor F(e_3), O(e_2, e_4) = F(e_2) \lor F(e_4)\].
Definition 17 [7]

The complement of a fuzzy soft set \((F, A)\) is denoted by \((F, A)^c\) and is defined by \((F, A)^c = (F^c, A)\), where \(F^c : \tilde{A} \rightarrow \tilde{P}(U)\) is a mapping given by \(F^c(\sigma) = (F(\neg \sigma))^c\) for all \(\sigma \in \tilde{A}\).

In this definition \(F^c\) is a mapping from the set \(\tilde{A}\) of not parameters to \(\tilde{P}(U)\). This definition of complement of a fuzzy soft set does not meet our requirements that complement of a set in classical sense really does. As such, we redefine the complement of a fuzzy soft set so that difficulties faced only due to the definition are removed.

Definition 18.

The complement of a fuzzy soft set \((F, A)\) is denoted by \((F, A)^c\) and is defined by \((F, A)^c = (F^c, A)\) where \(F^c : A \rightarrow \tilde{P}(U)\) is a mapping given by \(F^c(\alpha) = [F(\alpha)]^c\), \(\forall \alpha \in A\).

Example 5.

Let \(U = \{c_1, c_2, c_3, c_4\}\) be the set of four cars under consideration and \(E = \{e_1\text{ (costly)}, e_2\text{ (Beautiful)}, e_3\text{ (Fuel Efficient)}, e_4\text{ (Modern Technology), } e_5\text{ (Efficient)}\}\) be the set of parameters and \(A = \{e_1, e_2, e_3\} \subseteq E\).

Then \((F, A) = \{F(e_1) = \{(c_1, 0.7, 0), (c_2, 0.1, 0), (c_3, 0.2, 0), (c_4, 0.6, 0)\}\),

\(F(e_2) = \{(c_1, 0.8, 0), (c_2, 0.6, 0), (c_3, 0.1, 0), (c_4, 0.5, 0)\}\),

\(F(e_3) = \{(c_1, 0.1, 0), (c_2, 0.2, 0), (c_3, 0.7, 0), (c_4, 0.3, 0)\}\)\n
\(F^c(e_1) = \{(c_1, 1.0, 0.7), (c_2, 1.0, 0.1), (c_3, 1.0, 0.2), (c_4, 1.0, 0.6)\}\),

\(F^c(e_2) = \{(c_1, 1.0, 0.8), (c_2, 1.0, 0.6), (c_3, 1.0, 0.1), (c_4, 1.0, 0.5)\}\),

\(F^c(e_3) = \{(c_1, 1.0, 0.1), (c_2, 1.0, 0.2), (c_3, 1.0, 0.7), (c_4, 1.0, 0.3)\}\)\n
Our definition of complement of a fuzzy soft set leads to the following two propositions of exclusion and contradiction of a fuzzy soft set which are what we are indeed searching for. Of course it can be verified that these results are true for usual fuzzy sets where fuzzy reference function is 0.

Proposition 4.

For a fuzzy soft set \((F, A)\) over \(U\), we have,

1. \((F, A)^c = \tilde{A}\) (Exclusion)
2. \((F, A)^\circ = \tilde{\phi}\) (Contradiction)

Proof.

1. Let \((F, A) = \tilde{A}\) \(\Rightarrow (F, A)^c = (H, A)\), where \(\forall \varepsilon \in A\),

\[H(\varepsilon) = F(\varepsilon) \cup F^c(\varepsilon)\]

\[= \{\varepsilon, \mu_1(\varepsilon), \mu_2(\varepsilon) ; \varepsilon \in U\} \cup \{\varepsilon, \mu_1(\varepsilon), \mu_2(\varepsilon) ; \varepsilon \in U\}^c\]

\[= \{\varepsilon, \mu_1(\varepsilon), \mu_2(\varepsilon) ; \varepsilon \in U\} \cup \{\varepsilon, 1, \mu_1(\varepsilon) ; \varepsilon \in U\}\]

2. \((F, A)^\circ = \tilde{\phi}\) \(\Rightarrow (F, A)^c = (H, A)\), where \(\forall \varepsilon \in A\),

\[H(\varepsilon) = F(\varepsilon) \cap F^c(\varepsilon)\]

\[= \{\varepsilon, \mu_1(\varepsilon), \mu_2(\varepsilon) ; \varepsilon \in U\} \cap \{\varepsilon, \mu_1(\varepsilon), \mu_2(\varepsilon) ; \varepsilon \in U\}^c\]

\[= \{\varepsilon, \mu_1(\varepsilon), \mu_2(\varepsilon) ; \varepsilon \in U\} \cap \{\varepsilon, 1, \mu_1(\varepsilon) ; \varepsilon \in U\}\]

\[= \{\varepsilon, 1, \mu_1(\varepsilon) ; \varepsilon \in U\}\]

Thus \((F, A)^c = \tilde{A}\)
We see that in our definition, the following properties hold good for fuzzy soft sets $(F, A)$, $(G, B)$ and $(H, C)$ in a fuzzy soft class $(U, E)$.

1. (i) $(F, A) \triangledown (G, B) = (G, B) \triangledown (F, A)$
   
   (Commutativity)

2. (i) $(F, A) \triangledown (G, B) \triangledown (H, C)\ni (F, A) \triangledown (G, B) \triangledown (H, C)$
   
   (Associativity)

3. (i) $(F, A) \triangledown (G, B) \triangledown (H, C)$
   
   = $((F, A) \triangledown (G, B)) \triangledown (H, C)$

   (Distributivity)

4. (i) $(F, A) \triangledown (F, A) = (F, A)$
   
   (Idempotency)

5. $((F, A)^c)^c = (F, A)$

   (Involution)

Ahmad and Kharal [1] established the following results –

**Theorem 1 [1]**

For fuzzy soft sets $(F, A)$ and $(G, B)$ of a fuzzy soft class $(U, E)$, one has the following:

1. \[ ((F, A) \triangledown (G, B))^c \subseteq (F, A)^c \triangledown (G, B)^c \]

**Theorem 2 [1]**

For fuzzy soft sets $(F, A)$ and $(G, B)$ of a fuzzy soft class $(U, E)$, one has the following:

1. \[ (F, A)^c \triangledown (G, B)^c \subseteq ((F, A) \triangledown (G, B))^c \]

2. \[ (F, A) \triangledown (G, B) \triangledown (H, C)^c \subseteq (F, A) \triangledown (G, B)^c \]

One can verify that these results are valid from our standpoint.

It is well known that DeMorgan Law interrelate union and intersection via complements. Ahmad and Kharal [1] proved the following DeMorgan Laws. Here we show that the same results are true according to our new definition of complement of a fuzzy soft set.

**Theorem 3.**

For fuzzy soft sets $(F, A)$ and $(G, A)$ in a fuzzy soft class $(U, E)$, one has the following:

1. \[ (F, A) \triangledown (G, A)^c = (F, A)^c \triangledown (G, A)^c \]

2. \[ ((F, A) \triangledown (G, A))^c \subseteq ((F, A)^c \triangledown (G, A))^c \]

**Proof.**

1. Let $(F, A) \triangledown (G, A) = (H, A)$, where \( \forall \varepsilon \in A, H(\varepsilon) = F(\varepsilon) \cup G(\varepsilon) \)

Thus \( (F, A) \triangledown (G, A)^c = (F, A)^c \triangledown (H(\varepsilon))^c \)

Where \( \forall \varepsilon \in A, H(\varepsilon)^c = F(\varepsilon) \cap G(\varepsilon)^c \)

\[ = F(\varepsilon)^c \cap G(\varepsilon)^c \]

Again, \( (F, A)^c \triangledown (G, A)^c = (F, A)^c \triangledown (G, A)^c \)

where \( \forall \varepsilon \in A, I(\varepsilon) = F(\varepsilon)^c \cap G(\varepsilon)^c \)

Similarly \( (F, A)^c \triangledown (G, A)^c = (F, A)^c \triangledown (G, A)^c \)

2. Let $(F, A) \triangledown (G, A) = (H, A)$, where \( \forall \varepsilon \in A, H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon) \)

Thus \( (F, A) \triangledown (G, B)^c = (F, A)^c \cap (G, B)^c \)

\[ = (F(\varepsilon) \cup G(\varepsilon))^c \]

where \( \forall \varepsilon \in A, I(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)^c \)

Similarly \( (F, A) \triangledown (G, B)^c = (F, A)^c \triangledown (G, B)^c \)

\[ = (F(\varepsilon) \cap G(\varepsilon))^c \]
Again, \((F, A)^c \cup (G, A)^c = (F^c, A) \cup (G^c, A) = (I, A)\), say where
\[ \forall \varepsilon \in A, I(\varepsilon) = F^c(\varepsilon) \cup G^c(\varepsilon) \]
Thus \((F, A)^c \cup (G, A)^c = (F, A)^c \cup (G, A)^c\)

Next we show that the following De Morgan type of results are true in case of “AND” and “OR” operations of fuzzy soft sets according to our new notion of complement of a fuzzy soft set.

**Proposition 5.**
1. \( ((F, A) \cap (G, B))^c = (F, A)^c \cup (G, B)^c \)
2. \( ((F, A) \cup (G, B))^c = (F, A)^c \cap (G, B)^c \)

**Proof:**
1. Let \((F, A) \cap (G, B) = (H, A \times B)\), where \(H(\alpha, \beta) = F(\alpha) \cap G(\beta)\), \(\forall \alpha \in A\) and \(\forall \beta \in B\), where \(\cap\) is the operation ‘fuzzy intersection’ of two fuzzy sets.
   Thus \( ((F, A) \cap (G, B))^c = (H, A \times B)^c = (H^c, A \times B) \)
   where \(\forall (\alpha, \beta) \in A \times B, H^c(\alpha, \beta) = H(\alpha, \beta)^c = (F(\alpha) \cap G(\beta))^c = (F(\alpha))^c \cup (G(\beta))^c = F^c(\alpha) \cup G^c(\beta) \)

   Let \((F, A)^c \cup (G, B)^c = (F, A)^c \cup (G, B)^c = (O, A \times B), \) where \(O(\alpha, \beta) = F^c(\alpha) \cup G^c(\beta), \forall \alpha \in A\) and \(\forall \beta \in B\), where \(\cup\) is the operation ‘fuzzy union’ of two fuzzy sets.
   Thus \( ((F, A) \cap (G, B))^c = (F, A)^c \cup (G, B)^c \)
2. Let \((F, A) \cup (G, B) = (H, A \times B)\), where \(H(\alpha, \beta) = F(\alpha) \cup G(\beta)\), \(\forall \alpha \in A\) and \(\forall \beta \in B\), where \(\cup\) is the operation ‘fuzzy union’ of two fuzzy sets.
   Thus \( ((F, A) \cup (G, B))^c = (H, A \times B)^c = (H^c, A \times B) \)
   where \(\forall (\alpha, \beta) \in A \times B, H^c(\alpha, \beta) = H(\alpha, \beta)^c = (F(\alpha) \cup G(\beta))^c = (F(\alpha))^c \cap (G(\beta))^c = F^c(\alpha) \cap G^c(\beta) \)

Let \((F, A)^c \cap (G, B)^c = (F^c, A) \cap (G^c, B) = (O, A \times B), \) where \(O(\alpha, \beta) = F^c(\alpha) \cap G^c(\beta), \forall \alpha \in A\) and \(\forall \beta \in B\), where \(\cap\) is the operation ‘fuzzy intersection’ of two fuzzy sets.
Thus \( ((F, A) \cap (G, B))^c = (F, A)^c \cap (G, B)^c \)

We now give the definition of arbitrary fuzzy soft “AND” and “OR” as given below.

**Definition 19.**
Let \((F_1, A_1), (F_2, A_2), (F_3, A_3), \ldots, (F_n, A_n)\) be \(n\) fuzzy soft sets over the same universe \(U\). Then \((F_1, A_1) \cap (F_2, A_2) \cap (F_3, A_3) \cap \ldots \cap (F_n, A_n) = (H, A_1 \times A_2 \times A_3 \times \ldots \times A_n)\),
where \(H(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = F_1(\alpha_1) \cap F_2(\alpha_2) \cap F_3(\alpha_3) \cap \ldots \cap F_n(\alpha_n)\)
\(\forall \alpha_i \in A_i\), where \(\cap\) is the operation ‘fuzzy intersection’ of fuzzy sets.
Also \((F_1, A_1) \cup (F_2, A_2) \cup (F_3, A_3) \cup \ldots \cup (F_n, A_n) = (O, A_1 \times A_2 \times A_3 \times \ldots \times A_n)\),
where \(O(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = F_1(\alpha_1) \cup F_2(\alpha_2) \cup F_3(\alpha_3) \cup \ldots \cup F_n(\alpha_n)\)
\(\forall \alpha_i \in A_i\), where \(\cup\) is the operation ‘fuzzy union’ of fuzzy sets.

Next we show that the following De Morgan type of results are true in case of arbitrary fuzzy soft “AND” and “OR” according to our new notion of complement of a fuzzy soft set.

**Proposition 6.**
1. \( ((F_1, A_1) \cap (F_2, A_2) \cap (F_3, A_3) \cap \ldots \cap (F_n, A_n))^c = (F_1, A_1)^c \cap (F_2, A_2)^c \cap (F_3, A_3)^c \cap \ldots \cap (F_n, A_n)^c \)
2. \( ((F_1, A_1) \cup (F_2, A_2) \cup (F_3, A_3) \cup \ldots \cup (F_n, A_n))^c = (F_1, A_1)^c \cup (F_2, A_2)^c \cup (F_3, A_3)^c \cup \ldots \cup (F_n, A_n)^c \)

**Proof:**
1. Let \((F_1, A_1) \cap (F_2, A_2) \cap (F_3, A_3) \cap \ldots \cap (F_n, A_n) = (H, A_1 \times A_2 \times A_3 \times \ldots \times A_n)\)
Thus

\[
((F_1, A_1) \wedge (F_2, A_2) \wedge (F_3, A_3) \wedge \cdots \wedge (F_n, A_n))^c
\]

\[
= (H, A_1 \times A_2 \times A_3 \times A_n)^c
\]

\[
= (H^c, A_1 \times A_2 \times A_3 \times A_n)
\]

Where

\[
\forall (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in (A_1 \times A_2 \times A_3 \times \cdots \times A_n)
\]

\[
H^c (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)
\]

\[
= (F_1(\alpha_1) \cap F_2(\alpha_2) \cap F_3(\alpha_3) \cap \cdots \cap F_n(\alpha_n))^c
\]

\[
= (F_1(\alpha_1))^c \cup (F_2(\alpha_2))^c \cup (F_3(\alpha_3))^c \cup \cdots \cup (F_n(\alpha_n))^c
\]

Again let

\[
(F_1, A_1)^c \vee (F_2, A_2)^c \vee (F_3, A_3)^c \vee \cdots \vee (F_n, A_n)^c
\]

\[
= (F_1^c, A_1) \cup (F_2^c, A_2) \cup (F_3^c, A_3) \cup \cdots \cup (F_n^c, A_n)
\]

Where

\[
\forall (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in (A_1 \times A_2 \times A_3 \times \cdots \times A_n)
\]

\[
O(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)
\]

\[
= F_1^c (\alpha_1) \cup F_2^c (\alpha_2) \cup F_3^c (\alpha_3) \cup \cdots \cup F_n^c (\alpha_n)
\]

Thus the result follows immediately.

Proof of (2) can be established in a similar way.

Ahmad and Kharal [1] defined arbitrary fuzzy soft union and intersection in the following way.

Definition 20 [1]

Let \( \mathcal{S} = \{(F_i, A_i) \mid i \in I\} \) be a family of fuzzy soft sets in a fuzzy soft class \((U, E)\). Then one has the following.

\[
1. \tilde{\forall} (F_i, A_i)^c = \tilde{\forall} (F_i^c, A)
\]

\[
2. \tilde{\forall} (F_i, A_i)^c = \tilde{\forall} (F_i^c, A)^c
\]

We show that these results are true according to our new notions.

Proof

1. We have,

\[
\tilde{\forall} (F_i, A_i)^c = \tilde{\forall} (F_i^c, A)
\]

Where \( \forall \alpha \in A_i \),

\[
H(\alpha) = \cap F_i^c(\alpha) \ldots (1)
\]

Again suppose that

\[
\tilde{\forall} (F_i, A) = (I, A)\]. Then

\[
\tilde{\forall} (F_i, A) = (I, A)^c
\]

\[
= (I^c, A)
\]

where

\[
I^c(\alpha) = \bigcup_{i} F_i^c(\alpha) \ldots (2)
\]

From (1) and (2), we get the desired result.

2. We have

\[
\tilde{\forall} (F_i, A_i)^c = \tilde{\forall} (F_i^c, A) = (H, A)\]

where \( \forall \alpha \in A_i \),

\[
H(\alpha) = \cup F_i^c(\alpha) \ldots (1)
\]

Again suppose that

\[
\tilde{\forall} (F_i, A) = (I, A)\]. Then

\[
\tilde{\forall} (F_i, A) = (I, A)^c
\]

\[
= (I^c, A)
\]

where
\[ I'(\alpha) = \left[ I(\alpha) \right] = \left[ \bigcap_i F_i(\alpha) \right] \]

\[ \forall \alpha \in A, \text{ we have, } \quad I'(\alpha) = \left[ \bigcap_i F_i(\alpha) \right] = \bigcup_i F'_i(\alpha) \quad \text{(2)} \]

From (1) and (2), we get the desired result.

3. Conclusion

We have defined complement of a fuzzy set where fuzzy reference function is not always 0 and definition of the same for fuzzy sets with fuzzy reference function 0 can seen as particular cases of what we have done. Fuzzy soft sets, with our new notions can be seen more rational and logical than what we are having now. The definition of complement of a fuzzy soft set based on some wrong axioms have to be reintroduced and which is what we have done, thus arriving at the conclusion that the fuzzy soft sets, too, follow the classical set theoretic axioms of exclusion and contradiction in addition to all those properties that complement of a set in classical sense really does. We hope that our findings will help enhancing this study on fuzzy soft sets.

References