Efficient Quadrature Using Bernstein’s Polynomial Weights via Fusion of Two Dual-Perspectives

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Abstract— A new polynomial quadrature operator has been proposed which uses weight-functions of the well-known Bernstein’s Polynomial operator in its improved structure, achieved through a rather ingenious ‘Fusion’ of two dual perspectives. These weights are functions of the impugned variable of the unknown function being approximated, and are not mere constants. The new quadrature formula has been compared empirically with that quadrature using the well-known “Bernstein Operator”. The percentage absolute relative errors for the proposed quadrature formula and that with the “Bernstein Operator” have been computed for certain selected functions and with different number of node points in the interval of quadrature. It has been observed that the proposed quadrature formula produces exceedingly significantly better results.

Keywords—Dual-Fusion, Improved Structure; Percentage Absolute Relative Errors; Polynomial Quadrature; Weight Functions.

1. Introduction

The main fact motivating us to ponder-n-wonder as to whether or not we would be able to retain the simplicity of “equidistant points in the interval of quadrature” in our proposed quadrature formula, and still aspire successfully for its optimality, only through the manipulation of the ‘weight-function’. In fact we were successful. The next section details the ‘How’ part of it. And the section following that numerically illustrates that the potential of this new/proposed polynomial quadrature operator was so profound that it was almost the ‘Optimal’ one, inasmuch as it excelled over the well-known usual “Bernstein’s Polynomial quadrature Operator”, besides being exceedingly superior to that! The “Bernstein Operator” happened to be the foundational structure for the build-up of the proposed one, using weight-functions derived using a rather ingenious ‘Fusion’ of two dual perspectives. We are going to detail this in following section.

2. The Proposed Dual-Fusion Weighted Quadrature Operator

The problem of approximation arises in many contexts of ‘Numerical Analysis and Computing’. The ‘Quadrature’ is one such. Weirstrass (1885) proved his celebrated approximation theorem: if \( f \in C \left[ a, b \right] \); for every \( \delta > 0 \); there is a polynomial ‘\( p \)’ such that
\[
| f - p | < \delta.
\]
In other words, the result established the existence of an algebraic polynomial, in the relevant variable, capable of approximating the unknown function in that variable, as closely as we please.

The aforementioned result was a big beginning of the mathematicians’ interest in ‘Polynomial Approximation’ of an unknown function using its values generated experimentally or known otherwise at certain chosen ‘Knots’ of the domain of the relevant variable, as of interest to the scientist concerned. The Great Russian mathematician S. N. Bernstein proved the Weirstrass’ theorem in a manner which was very stimulating and interesting in many ways.

He first noted a simple but important fact that if the Weirstrass’ theorem holds for \( C \left[ 0, 1 \right] \), it also holds for \( C \left[ a, b \right] \) and holds conversely. Essentially \( C \left[ 0, 1 \right] \) and \( C \left[ a, b \right] \) are identical, for all practical purposes; as they are linearly isometric as normed spaces, order isomorphic as algebras (rings). Most important contribution in the Bernstein’s proof of this theorem consisted in the fact that Bernstein actually displayed a sequence of polynomials that approximate a given function \( f(x) \in C \left[ 0, 1 \right] \). If \( f(x) \) is any bounded function on \( C \left[ 0, 1 \right] \), the sequence of Bernstein’s Polynomials for \( f(x) \) is defined by:

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\[
(\text{Bn } (f))(x) = \sum_{k=0}^{n} W_{k,n} \ast f \left(\frac{k}{n}\right) \tag{2.0}
\]

Wherein, \( W_{k,n} = \binom{n}{k} x^k (1-x)^{n-k} \) are the respective weights for the values \( f \left(\frac{k}{n}\right) \) of the function at the knots \( \left\{ \frac{k}{n} \right\} [k = 0 \ (1) \ n] \).

Now, we are set to launch the “dual perspectives” seminal to the proposition of our novel ‘Polynomial Quadrature Operator’ in the same “Equi-Distant Knots” ‘Set-Up’ of Bernstein’s Operator, in what follows.

We, without any loss of generality as explained above, consider the approximation of an unknown function \( f(x); x \in C \ [0, \ 0.5] \). For such a case the usual ‘Bernstein Polynomial Approximation Operator’ in (2.0) boils down to:

\[
(\text{Bn } (f))(x) = \sum_{k=0}^{n} w_{k,n}(x) \ast f(\frac{k}{2n}) \tag{2.1}
\]

Wherein, \( w_{k,n}(x) = \binom{n}{k} (0.5-x)^k (0.5+x)^{n-k} \) are respective weights for values \( f(\frac{k}{2n}) \) of the function at the knots \( \left\{ k/(2n) \right\} [k = 0 \ (1) \ n] \).

Incidentally, as we could use a suitable transformation (translation-change of scale) of the variable “\( x \)”, we could assume, without loss of generality, that we are interested in the quadrature of a bounded function \( f(x); x \in C \ [0, \ 0.5] \). Imagine the line \( C \ [0, 1] \sim O1 \rightarrow O3 \) with the point \( O2 \), such that \( O1 \rightarrow O2: O2 \rightarrow O3 \sim 0.5 \cdot x: 0.5 + x; x \in C \ [0, 0.5] \). In the Primal sense, we could envisage “\( k \)” knots sitting on the line \( O1 \rightarrow O2 \), and “\( n-k \)” knots sitting on the line \( O2 \rightarrow O3 \). It is worth noting here that these “Primal Bernstein-Weights”; Say:

\[
P_{w1}(x; k, n) = \binom{n}{k} \left[ (0.5-x)^k (0.5+x)^{n-k} \right] \tag{2.2}
\]

Similarly, in a “dual” sense, we could have a visualization wherein the point \( x \ [0 \leq x \leq 0.5] \) divides the double-line in two parts of lengths \( (0.5 + x): (0.5 - x) \); \( O3 \rightarrow x: x \rightarrow O1 \). Thus, It is worth noting here that these “Dual Bernstein-Weights”; Say:

\[
P_{w2}(x; k, n) = \binom{n}{k} \left[ (0.5 + x)^k (0.5 - x)^{n-k} \right] \tag{2.3}
\]

Apparently, both \( P_{w2}(x; k, n) \) and \( P_{w1}(x; k, n) \), dual to each other, define a ‘Weight/ Probability Distribution’ each with the ‘\( n+1 \)’ knots \( \left\{ k/(2n) \right\} [k = 0 \ (1) \ n] \) being its support in \( C \ [0, 0.5] \).

As such, we define our ‘Optimal Weights’; Say \( W_{opt}(x; k, n) \), through a simple ‘Fusion’ of the two ‘Probability Distributions’ above in (2.2), and in (2.3) by taking their respective arithmetic mean. Therefore, we have:

\[
W_{opt}(x; k, n) = \left\{ \binom{n}{k} (0.5-x)^k (0.5+x)^{n-k} \right\} + \left\{ \binom{n}{k} (0.5 + x)^k (0.5-x)^{n-k} \right\} \] /2 \tag{2.4}

Thence, we propose a new formula which uses weight functions derived using a rather-ingenious ‘Fusion’ of two dual perspectives, as above. This ‘New Optimal Probabilistic Quadrature Operator’ happens to be as below.

Say; \( (\text{On } (f))(x) = \sum_{k=0}^{n} W_{opt}(x; k, n) \ast f(\frac{k}{2n}) \) \tag{2.5}

Wherein, \( W_{opt}(x; k, n) \) is as defined in (2.4).

At this point, we note that the original interval \( C \ [0, 1] \) is the domain of the impugned bounded function \( f(x) \).

Also, we note that the interval \( \left[ \frac{1}{2} - \frac{x}{2}, \frac{1}{2} \right] \) and \( \left[ \frac{1}{2}, \frac{1}{2} + \frac{x}{2} \right] \) are two sub-intervals of this domain.
It is significant to observe that, with \( x \in [0, 1/2] \), whereas the former sub-interval will grow to become \([0, \frac{1}{2}]\) for \( x=1/2 \), the later sub-interval will grow to become \([\frac{1}{2}, 1]\) for \( x=1/2 \). On the other hand, for \( x=0 \), both of them will degenerate to the point \( \frac{1}{2} \). In a sense, the two intervals are not only complimentary to each-other, but are dual to each-other.

It is interesting to note that if we use \( n \) [\( n \) being a positive integer] points/knots, beside ‘0’ on \([0, 1/2]\), our proposed operator, say \( (\text{On} (f)) (x) \) has ‘Zero-Error’ for \( \text{Quadrature of } f(x) = x^m \), \( \forall \ m \in [o(1)n] \), i.e. \( \forall \ m \leq n \).

Another point of curiosity would be to discover as to how our proposed operator \( (\text{On} (f)) (x) \) performs vis-à-vis the well-known ‘Bernstein’s Polynomial Quadrature Operator’. To accomplish that, we have taken FOUR example-functions in the following ‘Empirical Study’ section, in the sense of simulation. The aforesaid ‘Empirical Simulation Study’, as above has been accomplished using ‘MAPLE 13’. We have considered THREE illustrative values of ‘\( n \)’, i.e. \( n = 2, 4, \) and \( 6 \). The resultant values are tabulated in the relevant FOUR Tables, corresponding to the four respective illustrative functions.

### 3. Numerical Study

This section is also of prime interest, as herein we try to illustrate the potential of our proposed probabilistic operator \( (\text{On} (f)) (x) \). As apparent in the second section, as a prelude to our proposed operator, the mother-operators are the “Bernstein’s Polynomials for \( f \) (\( B_n \)(f) (x))” in (2.0). As such, we could not have an idea about cannot be conclusive about their relative supremacy in terms of better estimation potential; we have to discover their relative supremacy of efficient estimation only via a ‘Numerical Study’, as attempted in what follows in this section. In this simulated numerical study we have chosen four illustrative example-functions: \( \exp (x) \), \( 10^x \), \( \sin (2+x) \), and \( \ln (2+x) \). For simplicity of the numerical illustration we have confined to chosen illustrative \( n \)-values to be \( 2, 4, \) and \( 6 \).

We have considered numerical values (per the illustrative numerical study) of the “Percentage Relative Absolute Errors” in using the relevant operators by the evaluation of the expressions: namely \( \text{PRAbsErr} (\bullet) (\text{In } \%) \) for \( (\text{On} (f)) (x) \); & for \( (\text{Bn} (f)) (x) \), respectively →

\[
\left[ \int_{0}^{0.5} O_n(f; x) - \int_{0}^{0.5} f(x)dx \right] \times 100\% \left/ \left[ \int_{0}^{0.5} f(x)dx \right] \right. = \text{PRAbsErr} (O_n) ; \text{Say. And}
\]

\[
\left[ \int_{0}^{0.5} B_n(f; x) - \int_{0}^{0.5} f(x)dx \right] \times 100\% \left/ \left[ \int_{0}^{0.5} f(x)dx \right] \right. = \text{PRAbsErr} (B_n) ; \text{Say.}
\]

These “Percentage Relative Absolute Errors” [~ “\( \text{PRAbsErr} (\bullet) (\text{In } \%) \)"], calculated using the “MAPLE 13/Evaluation-Version” code, are tabulated in the following tables. This illustration has amply supported the fact that the “Percentage Absolute Errors” Numerical-Values” for our proposed probabilistic operator \( (\text{On} (f)) (x) \) are significantly lower than those for “Bernstein’s Polynomials for \( f \) (\( B_n \)(f) (x))” in (2.1).

#### Table 3.1: Percentage Relative Absolute Error [~ \( \text{PRAbsErr} (\bullet) (\text{In } \%) \)] of the Operators (In %) for Example-Function: \( f(x) = \exp(x) \).

<table>
<thead>
<tr>
<th>Operator</th>
<th>For ( n = 3 )</th>
<th>For ( n = 6 )</th>
<th>For ( n = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{PRAbsErr} (O_n) )</td>
<td>0.6922067613</td>
<td>0.3459837222</td>
<td>0.2306174110</td>
</tr>
<tr>
<td>( \text{PRAbsErr} (B_n) )</td>
<td>13.1778389900</td>
<td>12.8100393400</td>
<td>12.6858831600</td>
</tr>
</tbody>
</table>

#### Table 3.2: Percentage Relative Absolute Error [~ \( \text{PRAbsErr} (\bullet) (\text{In } \%) \)] of the Operators (In %) for Example-Function: \( f(x) = 10^x \).

<table>
<thead>
<tr>
<th>Operator</th>
<th>For ( n = 3 )</th>
<th>For ( n = 6 )</th>
<th>For ( n = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{PRAbsErr} (O_n) )</td>
<td>3.620482824</td>
<td>1.806981654</td>
<td>1.203602042</td>
</tr>
<tr>
<td>( \text{PRAbsErr} (B_n) )</td>
<td>32.2324619000</td>
<td>30.1619904700</td>
<td>29.4538021800</td>
</tr>
</tbody>
</table>
Table 3.3: Percentage Relative Absolute Error \(\text{PRAbsErr} (\bullet) \text{ (In %)}\) of the Operators (In %) for Example-Function: \(f(x) = \sin (2+x)\).

<table>
<thead>
<tr>
<th>Operator</th>
<th>For (n = 3)</th>
<th>For (n = 6)</th>
<th>For (n = 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PRAbsErr ((O_n))</td>
<td>0.6967080106</td>
<td>0.3484754609</td>
<td>0.2323558920</td>
</tr>
<tr>
<td>PRAbsErr ((B_n))</td>
<td>10.80028436000</td>
<td>10.46967437000</td>
<td>10.36074967000</td>
</tr>
</tbody>
</table>

Table 3.4: Percentage Relative Absolute Error \(\text{PRAbsErr} (\bullet) \text{ (In %)}\) of the Operators (In %) for Example-Function: \(f(x) = \ln (2+x)\).

<table>
<thead>
<tr>
<th>Operator</th>
<th>For (n = 3)</th>
<th>For (n = 6)</th>
<th>For (n = 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PRAbsErr ((O_n))</td>
<td>0.1710444528</td>
<td>0.08548673220</td>
<td>0.05697964897</td>
</tr>
<tr>
<td>PRAbsErr ((B_n))</td>
<td>6.72260019100</td>
<td>6.80339491600</td>
<td>6.82993707300</td>
</tr>
</tbody>
</table>

References