Fuzzy Tolerance Graphs
Sovan Samanta and Madhumangal Pal

Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University, Midnapore - 721 102, India.
email: mmpalvu@gmail.com

Abstract - In this paper, we define fuzzy tolerance, fuzzy tolerance graph, fuzzy bounded tolerance graph, fuzzy interval containment graph and regular representation of fuzzy tolerance graph, fuzzy unit tolerance graph and proper tolerance graph. Also some basic theorems related to the stated graphs have been presented.

Keywords: Fuzzy graphs, intersection graphs, fuzzy tolerance graphs.

1. Introduction

In mathematical area of graph theory, an intersection graph is a graph that represents the pattern of intersection of family of sets. An interval graph is the intersection of multiset of intervals on real line. Interval graphs are useful in resource allocation problem in operations research. Besides, interval graphs are used extensively in mathematical modeling, archaeology, developmental psychology, ecological modeling, mathematical sociology and organization theory.

Tolerance graphs [5] are generalization of interval graphs in which each vertex can be represented by an interval and a tolerance such that an edge occurs if and only if the overlap of corresponding intervals is at least as large as the tolerance associated with one of the vertices. Hence a graph \( G = (V, E) \) is a tolerance graph [5] if there is a set \( I = \{I_x: x \in V\} \) of closed real intervals and a set \( T = \{T_y: y \in V\} \) of positive real numbers such that \((x, y) \in E \text{ if } |I_x \cap I_y| \geq \min\{T_x, T_y\}\). The collection \(<I, T>\) of intervals and tolerances is called tolerance representation of the graph \( G \).

Tolerance graphs were introduced in order to generalize some well known applications of interval graphs. The main motivation was to model resource allocation and certain scheduling problems, in which resources, such as rooms and vehicles, can tolerate sharing among users. Tolerance graphs find in a natural way for applications in biology and bioinformatics. The tolerance graphs find numerous other applications in constrained temporal reasoning, data transmission through networks to efficiently scheduling aircraft and crews, as well as contributing to genetic analysis and studies of the brain.

1.1 Fuzzy sets

The fuzzy systems have been used with success in last years, in problems that involve the approximate reasoning. The objective of the work is to specify the fuzzy systems with the addition of the interval theory in its components. Fuzzy graph theory was introduced by Azriel Rosenfeld [16] in 1975. Though it is very young, it has been growing fast and has numerous applications in various fields.

A fuzzy set \( A \) on a set \( X \) is characterized by a mapping \( m: X \rightarrow [0,1] \), called the membership function. We shall denote a fuzzy set as \( A = (X, m) \). The support of \( A \) is \( \text{supp } A = \{x \in X \mid m(x) \neq 0\} \). The core of \( A \) is the crisp set of all members whose membership values are \( 1 \). \( A \) is non trivial if \( \text{supp } A \) is nonempty. The height of \( A \) is \( h(A) = \max\{m(x) \mid x \in X\} \). \( A \) is normal if \( h(A) = 1 \). The membership function of the intersection of two fuzzy sets \( A \) and \( B \) with membership functions \( m_A \) and \( m_B \) respectively is defined as the minimum of the two individual membership functions. \( m_{A \land B} = \min(m_A, m_B) \). Let \( A = (X, m) \) and \( B = (X, m') \) be two fuzzy sets on \( X \) then \( A \leq B \) (fuzzy subset) if \( m(x) \leq m'(x) \) for all \( x \in X \). The family of all fuzzy subsets is denoted by \( \Phi \). A fuzzy set \( A = (X, m_A) \) is said to be convex if its membership value satisfies the following condition \( m_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{m_A(x_1), m_A(x_2)\} \) for any \( x_1, x_2 \in X \) and \( \lambda \in [0,1] \).
A fuzzy number is a real fuzzy interval. In a fuzzy graph an arc \((x, y)\), we define the strength iff:

\[
\mu(x, y) = \sum_{i=1}^{n} \sigma(x) \cap \sigma(y). \]

So \(\mu\) is a fuzzy relation on \(\sigma\). A fuzzy diagraph \(\tilde{G} = (\sigma, \mu)\) is similarly defined except \(\tilde{\mu} : V \times V \to [0, 1]\) need not be symmetric. We call the pairs \(((x, y), \tilde{\mu}(x, y))\) fuzzy arcs to emphasize that the symmetry is not required. McAllister uses a measure of fuzziness of the fuzzy intersection of two fuzzy sets to define two structures which together are called a fuzzy intersection graph. The strength of connectedness between two vertices \(u\) and \(v\) is

\[
\mu^\sigma(u, v) = \sup\{\mu^k(u, v) | k = 1, 2, \ldots\} \text{ where } \mu^k(u, v) = \sup\{\mu(u, u_i) \land \mu(u_i, u_{i+1}) \land \ldots \land \mu(u_{i+k}, v) | u_i, u_{i+1}, \ldots, u_{i+k} \in V\}.
\]

An edge \((u, v)\) is a fuzzy bridge of \(\tilde{G}\) if deletion of \((u, v)\) reduces the strength of connectedness between the pair of vertices. A vertex \(u\) is a fuzzy cut vertex of \(\tilde{G}\) if deletion of \(u\) reduces the strength of connectedness between other pair of vertices. The order of fuzzy graph \(\tilde{G}\) is \(O(\tilde{G}) = \sum_{u \in V} |\sigma(u)|\). The size of fuzzy graph \(\tilde{G}\) is \(S(\tilde{G}) = \sum_{\mu(u, v) > 0} \mu(u, v)\). In a fuzzy graph an arc \((u, v)\) is said to be strong arc or strong edge, if \(\mu(u, v) \geq \mu^\sigma(u, v)\) and the node \(v\) is said to be strong neighbour of \(u\). If \((u, v)\) is not strong arc then \(v\) is called isolated node or isolated vertex.

Given \(t \in [0, 1]\) and a fuzzy set \(A\), we define the \(t\)-cut level set of \(A\) to be the crisp set \(A' = \{x \in supp A | m(x) \geq t\}\). The \(t\)-cut level graph of \(\tilde{G}\) is the crisp graph \(\tilde{G}' = (\sigma', \mu')\). For a family \(\Phi\) of fuzzy subsets, the \(t\)-cut level family of \(\Phi\) is

\[
\Phi' = \{A' | A \in \Phi\}. \]

The fuzzy intersection (min operator) is used to define the fuzzy intersection graph of a family of fuzzy sets. As in common to fuzzy set theory a crisp set is identified with a characteristic function. When context is clear two concepts are used interchangeably. Crisp definitions are extended using the concept of cut level sets. Significant results are discussed between fuzzy structure and the corresponding cut level sets.

Fuzzy intersection graph [7] is defined as: \(\Phi = \{A_i = (X, m_i), A_2 = (X, m_2), \ldots, A_n = (X, m_n)\}\) be a finite family of fuzzy sets defined on a set \(X\) and consider \(\Phi\) as crisp vertex set \(V = \{v_1, v_2, \ldots, v_n\}\). The fuzzy intersection graph of \(\Phi\) is the fuzzy graph \(Int(\Phi) = (V, \sigma, \mu)\) where \(\sigma : V \to [0, 1]\) is defined by \(\sigma(v_i) = h(A_i)\) and \(\mu : V \times V \to [0, 1]\) is defined by

\[
\mu(v_i, v_j) = \begin{cases} 
\begin{array}{ll} 
h(A_i \cap A_j), & \text{if } i \neq j \\
0, & \text{if } i = j 
\end{array} 
\end{cases}
\]

Here \(E = \{\{v_i, v_j \} \mid \mu(v_i, v_j) > 0\}\) is called the edge set of the fuzzy graph. An edge \((v_i, v_j)\) has a zero strength if \(m_i \cap m_j\) is zero function (empty intersection). A fuzzy interval graph [4] is the fuzzy intersection graph of a finite family of fuzzy intervals.

### 1.3 Review of Previous Works


1.4 Our works

In this paper we define fuzzy tolerances, then fuzzy tolerance graphs and prove some basic results.

2 Fuzzy Tolerance Graphs

2.1 Fuzzy Tolerances

Definition 1 Fuzzy tolerance of a fuzzy interval is denoted by $T$ and defined by an arbitrary fuzzy interval whose core length is a positive real number. If the real number is taken $L$ and $|i_k - i_{k-1}| = L$ where $i_{k-1}, i_k \in R$ then fuzzy tolerance is a fuzzy set of the interval $[i_{k-1}, i_k]$.

The core and support of a fuzzy tolerance are defined as tolerance core and tolerance support respectively. The tolerance core and tolerance support of fuzzy tolerance $T$ are denoted by $c(T)$ and $s(T)$ respectively. Fuzzy tolerance may be a fuzzy number. The fuzzy tolerance is shown in Figure 1.

![Figure 1: Fuzzy tolerance](image)

2.2 Fuzzy tolerance graphs

Fuzzy tolerance graph is the generalization of fuzzy interval graph which is defined below.

Definition 2 Let $I = \{I_1, I_2, \ldots, I_n\}$ be a finite family of fuzzy intervals on the real line and $T = \{T_1, T_2, \ldots, T_n\}$ be the corresponding fuzzy tolerances. We consider $I$ as a crisp vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The fuzzy tolerance graph is the fuzzy graph $(V, \sigma, \mu)$ where $\sigma : V \to [0, 1]$ is defined by $\sigma(v_i) = h(I_i) = 1$ for all $v_i \in V$ and $\mu : V \times V \to [0, 1]$ is defined by

$$
\mu(v_i, v_j) = \begin{cases}
1, & \text{if } c(I_i \cap I_j) \geq \min\{c(T_i), c(T_j)\} \\
\frac{s(I_i \cap I_j) - \min\{s(T_i), s(T_j)\}}{s(I_i \cap I_j)}, & \text{ otherwise if } s(I_i \cap I_j) \geq \min\{s(T_i), s(T_j)\} \\
0, & \text{ otherwise.}
\end{cases}
$$

where $c(I_i \cap I_j)$ is the core of the intersection of the intervals $I_i$ and $I_j$. $\min\{c(T_i), c(T_j)\}$ is the minimum of the cores of the corresponding tolerances $T_i$ and $T_j$. Also $s(I_i \cap I_j)$, $\min\{s(T_i), s(T_j)\}$ are the support of the intersection of intervals $I_i, I_j$ and minimum support the of tolerances $T_i, T_j$ respectively.

If the fuzzy tolerances are not incorporated with the fuzzy intervals, the corresponding graph becomes
fuzzy interval graph. The fuzzy tolerance graph have been shown in Figures 2, 3, 4 in three different cases. We consider two fuzzy intervals $I_i, I_j$ together with two fuzzy tolerances $T_i, T_j$. In the first case minimum of two tolerance cores $\min\{c(T_i), c(T_j)\}$ is less than the intersection core $c(I_i \cap I_j)$ of the intervals $I_i$ and $I_j$. In this case, edge membership value is 1. If $\min\{c(T_i), c(T_j)\}$ is greater than $c(I_i \cap I_j)$ then second case arises where membership value of the edge is less than 1. Here minimum of two tolerance supports $\min\{s(T_i), s(T_j)\}$ is less than the intersection support $s(I_i \cap I_j)$ of the intervals $I_i, I_j$. Otherwise, the third case arises where edge membership value is 0.

If $<I, T>$ is a tolerance representation of a crisp graph $G$ and $v$ is a point instead of an interval then the corresponding vertex is isolated vertex of $G$. But it may not be true in the fuzzy sense. If $<I, T>$ is a fuzzy tolerance representation of $\xi$ and $f$ is a fuzzy number instead of fuzzy interval, then the corresponding vertex may not be isolated. Intersection of supports of a fuzzy interval and a fuzzy number may be greater than minimum support of their corresponding tolerances. So there is an edge of some positive strength. If it is not strong arc then the fuzzy number is isolated otherwise it is not isolated.

![Figure 2: Fuzzy tolerance graph with edge membership value 1](image)

![Figure 3: Fuzzy tolerance graph with edge membership value less than 1](image)
Example 1 Let us consider an example of three fuzzy intervals $I_1$, $I_2$, $I_3$ with corresponding fuzzy tolerances $T_1$, $T_2$, $T_3$. Let the core and support of $I_1$ be $[2,7]$ and $[1,8]$; that of $I_2$ are $[4,8]$ and $[2.5,9.5]$; that of $I_3$ are $[9,13]$ and $[6.5,15.5]$. The core and support lengths of $T_1$, $T_2$, $T_3$ be respectively $\{1,1.2\}$, $\{0.8,1\}$, $\{0.5,2\}$. So $h(I_1 \cap I_2) = 1$, $h(I_2 \cap I_3) = 0.75$, and $h(I_1 \cap I_3) = 0.43$. Clearly, the edge membership values of the edges $(v_1, v_2)$, $(v_2, v_3)$ and $(v_1, v_3)$ are obtained as $1$, $0.5$ and $0.08$ respectively. The corresponding diagram is shown in Figure 5.

![Figure 4: Fuzzy tolerance graph with edge membership value 0](image1)

2.3 Fuzzy bounded tolerance graphs

If a crisp graph $G$ has a tolerance representation $< I, T >$, such that $T_j | I_j |$ for every $j = 1, 2, \ldots, n$ then $G$ is called bounded tolerance graph [8].

Now we define fuzzy bounded tolerance graph as below.

Definition 3 Let $I = \{ I_1, I_2, \ldots, I_n \}$ be a finite family of fuzzy intervals on real line and $T = \{ T_1$, $T_2, \ldots, T_n \}$ be the corresponding fuzzy tolerances. If fuzzy interval $I_1$ with core
A fuzzy bounded tolerance representation then no fuzzy interval $I_v$, $v \in V$ is fuzzy number.

**Theorem 1** If $\xi$ is a fuzzy interval graph then $\xi$ is a fuzzy tolerance graph with constant core and constant support of tolerances.

**Proof.** Let $\xi$ be a fuzzy interval graph with a representation $I_v$ assigned to the vertex $v$. Let the cores of fuzzy intervals $I_x$ and $I_y$ be denoted by $c(I_x)$ and $c(I_y)$ and that of supports be $s(I_x)$ and $s(I_y)$. Also $c(I_x \cap I_y) = (c(I_x) \cup c(I_y))$ and $s(I_x \cap I_y) = s(I_x) \cup s(I_y)$. Let $k_1$ and $k_2$ be positive real numbers such that $k_1 < |c(I_x \cap I_y)|$ and $k_2 < |s(I_x \cap I_y)|$ for all $x, y \in V$ with $k_1 \leq k_2$.

Thus the intervals $\{I_v \mid v \in V\}$ together with tolerances with core $k_i$ and support $k_2$ give a fuzzy tolerance representation $\Omega$.

**Theorem 2** If $\xi$ is a fuzzy tolerance graph with constant core and constant support of tolerances then $\xi$ is fuzzy bounded tolerance graph.

**Proof.** Let $\xi = \{I_1, I_2, \ldots, I_n\}$ be a finite family of fuzzy intervals on real line and $T = \{T_1, T_2, \ldots, T_n\}$ be the corresponding fuzzy tolerances. Let $\xi$ be a fuzzy tolerance graph with $I$ and $T$ be the corresponding fuzzy interval and fuzzy tolerance representation. Let $k_1$ and $k_2$ be two positive real numbers. Let $c(T_i) = k_1$ and $s(T_i) = k_2$ for $i = 1, 2, \ldots, n$.

If $c(I_i) \geq k_1$ and $s(I_i) \geq k_2$ for all $i = 1, 2, \ldots, n$ then $\xi$ is fuzzy bounded tolerance graph. If $c(I_i) < k_1$ for any $i$ and $s(I_i) < k_2$ for any $j$ then we take $c(I_i) = k_1$ and $s(I_i) = k_2$, to make $\xi$ be bounded. In this case no adjacencies will change for the vertices.

**Theorem 3** Let $\xi$ be a fuzzy tolerance graph. For each $t \in [0, 1]$, the cut level graphs are tolerance graph.

**Proof.** Let $\xi$ be a fuzzy tolerance graph and let $\xi = \{I_1, I_2, \ldots, I_n\}$ be a finite family of fuzzy intervals on real line and $T = \{T_1, T_2, \ldots, T_n\}$ be the corresponding fuzzy tolerances. For each $t \in [0, 1]$, $I_i ^t \in I_i ^t$ for all $i = 1, 2, \ldots, n$. So it is a crisp interval. Also $t \in [0, 1]$ then $T_i ^t \in T_i ^t$ is crisp length that is a crisp tolerance.

So crisp intervals together with crisp tolerances represent a tolerance representation. Hence $\xi ^t$, $t \in [0, 1]$ is a tolerance graph.

Weakly chordal graphs [5] are those graphs with no induced subgraphs isomorphic to $C_n$ or $\overline{C_n}$ for $n \geq 5$. A graph $G$ is called alternately orientable graphs [5] if this is an orientation of $G$ which is transitive on every chordless cycle of length greater than or equal to 4, i.e., the directions of the oriented edges must alternate. A graph $G$ is perfect if for all induced subgraphs $H$ of $G$, the chromatic number of $H$ equals the number of vertices in the largest clique in $H$. Now we prove the following result.

**Theorem 4** If $\xi$ is a fuzzy tolerance graph then each cut level graph is (i) weakly chordal, (ii) alternately orientable, (iii) perfect graph.
Proof. We know that each cut level graph of fuzzy tolerance graph is a tolerance graph. Also we know that tolerance graphs are (i) weakly chordal, (ii) alternately orientable, (iii) perfect graph [5]. So each cut level graph of fuzzy tolerance graphs is (i) weakly chordal, (ii) alternately orientable, (iii) perfect graph. \( \Omega \)

3 Fuzzy Interval Containment Graphs

An interval containment graph [5] \( G = (V,E) \) is one that can be represented by a set of real intervals \( I = \{ I_i : v \in V \} \) so that \( (k,l) \in E(G) \) precisely when one of \( I_k, I_l \) contains the other. Such a representation is called an interval containment representation.

Now we define fuzzy interval containment graph as follows.

**Definition 4** Let \( I = \{ I_1, I_2, \ldots, I_n \} \) be a finite family of fuzzy intervals on real line. We consider the crisp vertex set \( V = \{ v_1, v_2, \ldots, v_n \} \). The fuzzy interval containment graph is the fuzzy graph \( (V, \sigma, \mu) \) where \( \sigma : V \to [0,1] \) is defined by \( \sigma(v_i) = h(I_i) = 1 \) for all \( i = 1, 2, \ldots, n \) and \( \mu : V \times V \to [0,1] \) is defined by

\[
\mu(v_i, v_j) = \begin{cases} 
1, & \text{if core and support of one of } I_i, I_j \text{ contain the other} \\
\frac{c(I_i \cap I_j)}{2 \min \{c(I_i), c(I_j)\}} + \frac{s(I_i \cap I_j)}{\min \{s(I_i), s(I_j)\}}, & \text{otherwise.}
\end{cases}
\]

where \( c(I_i \cap I_j), s(I_i \cap I_j) \) are the core and support of the intersection of the intervals \( I_i, I_j \) and \( c(I_i), c(I_j) \), \( s(I_i), s(I_j) \) are the minimum of core and support of the intervals \( I_i \) and \( I_j \).

**Example 2** Let us consider an example of four fuzzy intervals \( I_1, I_2, I_3, I_4 \). Let the core and the support of \( I_1 \) be \([2.7] \) and \([1.8] \); that of \( I_2 \) be \([4.6] \) and \([3.7] \); that of \( I_3 \) be \([9.13] \) and \([7.5,14.5] \); that of \( I_4 \) be \([12,16] \) and \([11,17] \). So \( h(I_1 \cap I_2) = 1, h(I_3 \cap I_4) = 1 \) and \( h(I_1 \cap I_3) = 0.2 \). Clearly, the edge membership values of the edges \((v_1, v_2), (v_1, v_3)\) and \((v_2, v_4)\) are respectively \( 1, 0.007 \) and \( 0.42 \). The diagramatic representation of this example is given in Figure 6.

![Figure 6: Fuzzy interval containment graph](image)

**Theorem 5** If \( \xi \) is a fuzzy interval containment graph with edge membership value \( 1 \) or \( 0 \) then \( \xi \) has a fuzzy tolerance representation with core of an interval equals to core of corresponding tolerance and support of the interval equals to support of the corresponding tolerance.
Proof. Let $I=\{I_1, I_2, \ldots, I_n\}$ be a finite family of fuzzy intervals on real line. We consider the corresponding crisp vertex set $V=\{v_1, v_2, \ldots, v_n\}$. Let $\xi= (V, \sigma, \mu)$ be a fuzzy interval containment graph of the fuzzy intervals.

Let $\mu(v_i, v_j) = 1$ in $\xi$. Then core and support of one interval contain the other. So intersection of the two cores equals to minimum of two cores. Let $\xi'= (V, \sigma', \mu')$ be a fuzzy tolerance graph of $I$ together with fuzzy tolerances $T=\{T_1, T_2, \ldots, T_n\}$ such that cores of the fuzzy tolerances equal to cores of intervals and supports of the fuzzy tolerances equal to the supports of the intervals. So $c(I_1 \cap I_2)= \min(c(I_1), c(I_2)) = \min(c(T_1), c(T_2))$. Then $\mu'(v_i, v_j) = 1$ in $\xi'$.

Let $\mu(v_k, v_i) = 0$ in the graph $\xi$. Then there is no common part of cores and supports. Now in $\xi'$, $c(I_k \cap I_1)= 0$ and $s(I_k \cap I_1)= 0$. So $\mu(v_k, v_i) = 0$ in $\xi'$.

Hence if $\xi$ is a fuzzy interval containment graph with edge membership value $1$ or $0$ then $\xi$ has a fuzzy tolerance representation with cores of fuzzy intervals equal to cores of corresponding tolerances and supports of the intervals equal to supports of the tolerances. $\Omega$

If $\xi$ is a fuzzy tolerance graph with a fuzzy tolerance representation with core of an interval equals to core of corresponding tolerance and support of the interval equals to support of the corresponding tolerance then it is clear that $\xi$ has a fuzzy interval containment graph representation using same fuzzy intervals.

4 Regular representation of fuzzy tolerance graphs

A fuzzy tolerance graph is said to have regular representation if it satisfies the following three properties

(1) Any fuzzy tolerance core and support length larger than the core and support lengths of the corresponding fuzzy interval can be set to infinity without changing the adjacencies of vertices in the fuzzy tolerance graph.

(2) All core and support length of fuzzy tolerances are distinct.

(3) No two different fuzzy interval cores and supports share an end point.

Theorem 6 Every fuzzy tolerance graph has a regular representation.

Proof. Let $I=\{I_1, I_2, \ldots, I_n\}$ be a finite family of fuzzy intervals on real line and $T=\{T_1, T_2, \ldots, T_n\}$ be the corresponding fuzzy tolerances. Considering $I$ as a crisp vertex set $V=\{v_1, v_2, \ldots, v_n\}$, let $\xi= (V, \sigma, \mu)$ be a fuzzy tolerance graph and $<I, T>$ be the tolerance representation of it. Let $L(c(I_1))$ and $R(c(I_1))$ be the left and right end points of the core $c(I_1)$ and that of support $s(I_1)$ be $L(s(I_1))$ and $R(s(I_1))$ respectively for each $v \in V$.

For a vertex $v_i$, let $c(T_i)=c(I_i)$ and $s(T_i)=s(I_i)$. There exists an edge $(v_i, v_j)$ in $\xi$ if $c(I_i \cap I_j) \geq \min(c(T_i), \min(c(T_1)), \min(c(T_2)), \ldots, \min(c(T_n)))$. Otherwise if $c(I_i \cap I_j) \geq \min(s(I_1), s(I_2), \ldots, s(I_n))$, it is $s(I_i \cap I_j)$. So tolerance core and support for $v_i$ can be set to infinity without changing the adjacencies of vertices in the fuzzy tolerance graph. Thus, any fuzzy tolerance core and support length larger than the core and support length of the corresponding fuzzy interval can be set to infinity without changing the adjacencies of vertices in the fuzzy tolerance graph.

Let $\epsilon$ be the smallest positive number appearing in the union of the sets

(i) $\{L(c(I_i)) - L(c(I_j))\}$, (ii) $\{R(c(I_i)) - R(c(I_j))\}$,

(iii) $\{L(c(I_i)) - R(c(I_j))\}$, (iv) $\{c(T_i)\}$, (v) $\{c(T_i) - c(T_j)\}$,

(vi) $\{c(T_i) - c(I_i)\}$ .

If $x$ and $y$ be two distinct vertices with $c(T_x) = c(T_y)$, choosing one of them, say $x$, replacing $c(T_y)$ by $c(T_y) = c(T_x) - \frac{\epsilon}{2}$, we leave $c(T_y)$ unchanged. We show that this gives a representation of $\xi$ with one fewer repeated tolerance core. If $(x, z) \in E$,
then $|c(I_x) \cap c(I_z)| \geq \min \{c(T_x), c(T_z)\}$ is one possible case. If $(x, z) \notin E$, then $|c(I_x) \cap c(I_z)| < \min \{c(T_x), c(T_z)\}$ is a possible case and by our choice of $\varepsilon$ we know $\min \{c(T_x), c(T_z)\} - |c(I_x) \cap c(I_z)| \geq \varepsilon$.

Thus $|c(I_x) \cap c(I_z)| \leq \min \{c(T_x), c(T_z)\} - \varepsilon < \min \{c(T_x), c(T_z)\}$ as desired. If necessary, we recompute $\varepsilon$ and repeat the process until all tolerance core are distinct. The similar process can be done for the case of tolerance support.

Now suppose two different fuzzy intervals share an end point of core. Let $S = \{L(c(I_x)), R(c(I_x)) \forall v \in V\}$ be the set of core end points. Let $x$ and $y$ be the distinct elements of $V$ for which $p$ is an end point of $c(I_x)$ and $c(I_y)$. If there exist $x \in V$ with $R(c(I_x)) = p$, picking the one whose length $c(I_y)$ is longest and replacing $c(I_x)$ by $c(I_y) = [L(c_x) + \varepsilon, R(c_y) + \varepsilon]$ otherwise, we pick $x \in V$ with $L(c(I_x)) = p$ and $|c(I_y)|$ as large as possible and replace $c(I_x)$ by $c(I_x) = [L(c(I_x)) + \varepsilon, R(c(I_x)) + \varepsilon]$.

It is not hard to see that this gives a representation of $\xi$ with one fewer pair of elements sharing an end point. All tolerances are still distinct. If necessary, we recompute $\varepsilon$ and repeat the process until all end points of core are distinct. The similar process can be applied to support also.

\[\Omega\]

5 Fuzzy unit tolerance graph and fuzzy proper tolerance graphs

A unit tolerance graph [3] is one that has a tolerance representation in which all intervals have the same (unit) length and proper tolerance graph [3] is one that has a tolerance representation in which no interval is properly contained in another. First we define fuzzy unit interval graph and fuzzy proper interval graph then that in tolerance representation.

Definition 5 A fuzzy interval graph $\xi$ that has a representation in which all fuzzy intervals have same core lengths and same support lengths is called fuzzy unit interval graph. Similarly, if $\xi$ has a representation in which no fuzzy interval core and support properly contain another fuzzy interval core and support, $\xi$ is called fuzzy proper interval graph.

Definition 6 A fuzzy unit tolerance graph is a fuzzy tolerance graph that has tolerance representation in which all fuzzy interval core lengths are same and the support lengths are same. A fuzzy proper tolerance graph is one that has tolerance representation in which no fuzzy interval core and support properly contain another fuzzy interval core and support.

Clearly the class of fuzzy unit tolerance graph is a subset of the class of fuzzy proper tolerance graph. We now give an example of fuzzy graph and its fuzzy unit interval representation.

Example 3 Let us consider six fuzzy intervals with corresponding tolerances. The cores and supports of fuzzy intervals are given respectively as $I_a[5,11][4,12]; I_e[10,16][9,17]; I_c[3,9][2,10]; I_d[6,12][5,13]; I_e[1,7][0,8]; I_f[8,14][7,15]$. The lengths of cores and supports of the corresponding tolerances are given as $T_a \{6,8\}, T_b \{1,2\}, T_c \{1,2\}, T_d \{6,8\}, T_e \{1,2\}, T_f \{4,6\}$. Here core lengths of the fuzzy intervals are same ($=6$) and support lengths are same ($=8$) for all intervals. So the fuzzy intervals together with fuzzy tolerances represent fuzzy unit tolerance graph. The diagram is shown in Figure 7.
Theorem 7  Any fuzzy unit or proper tolerance representation may be assumed to have fuzzy bounded tolerances.

Proof. We fix a fuzzy unit or proper tolerance representation $< I, T >$ of a fuzzy unit or proper tolerance graph. We may assume all endpoints of core and support in this representation are distinct. We replace $c(\uparrow x)$ by $|c(\uparrow x)|$ for each $x \in V$ when $c(T_x) \geq |c(I_x)|$ and $s(T_x)$ replace by $|s(I_x)|$ for each $x \in V$ when $s(T_x) \geq |s(I_x)|$, Since there are no containments of core and support of fuzzy intervals, this will not change any adjacencies.

References


Madhumangal Pal is a Professor of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, India. He received University Silver Medal for rank second in B.Sc. (Honours) in the year 1988 from Vidyasagar University, India. He received University Gold Medal for rank first in M.Sc. in the year 1990 from the same university. He received Computer Division medal from Institution of Engineers (India) in the year 1996 for the best research work published in the Institution journal jointly with Prof. G.P.Bhattacharjee. He is also Editor-in-Chief of Journal of Physical Sciences. He is member of Editorial Board of International Journal of Computer Sciences, Systems Engineering and Information Technology, International Journal of Fuzzy Systems & Rough Systems, Advanced Modeling and Optimization, Romania International Journal of Logic and Computation, Malaysia. He is a reviewer of several international journals. He has written several books on Mathematics and Computer Science. His research interest includes computational graph theory, fuzzy matrices, game theory and regression analysis, parallel and genetic algorithms, etc.

Sovan Samanta received his B.Sc. degree in 2007 and M.Sc. degree in 2009 in Applied Mathematics from Vidyasagar University, India. He is now currently a research scholar in the Department of Applied Mathematics, Vidyasagar University since 2010. His research interest includes fuzzy graph theory.