Further Acceleration of the Newton-Ostrowski Method for Solving Nonlinear Equations

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Abstract - A family of four-point iterative methods for solving nonlinear equations is constructed using a suitable parametric function and three arbitrary real parameters. It is proved that these methods have the convergence order of nine to sixteen. Per iteration the new methods requires four evaluations of the function and one evaluation of its first derivative. We obtain the optimal order of convergence which supports the Kung and Traub conjecture. The Kung and Traub conjectured that the multipoint iteration methods, without memory based on \( n \) evaluations could achieve optimal convergence order \( 2^{n-1} \). Thus, we present a new method which agrees with Kung and Traub conjecture for \( n = 5 \). We shall examine the effectiveness of the new Newton-Ostrowski methods by approximating the simple root of a given nonlinear equation.

Keywords: Newton-Ostrowski methods; nonlinear equations; optimal order of convergence; Computational efficiency; Kung-Traub conjecture.

Subject Classifications: AMS (MOS): 65H05.

1. Introduction

In this paper, we present a new family of the Newton-Ostrowski methods to find a simple root \( \alpha \) of the nonlinear equation

\[
f(x) = 0,
\]

(1)

Where \( f: I \subset \mathbb{R} \rightarrow \mathbb{R} \) is a scalar function on an open interval \( I \) and it is sufficiently smooth in a neighbourhood of \( \alpha \). It is well known that the techniques to solve nonlinear equations have many applications in science and engineering. In order to construct the new sixteenth-order method we utilise three well known methods, namely the classical Newton method for its simplicity and its second-order of convergence [4,9,10,15], the original Ostrowski method with fourth-order convergence [10] and the recently introduced the Newton-type eighth-order method [1,2,14]. It is well established that the Ostrowski method has been previously improved from 5 to 7, [3,6,7,12].

The new family of the Newton-Ostrowski methods requires four evaluations of the function and one evaluation of its first derivative. Hence the new methods have a better efficiency index than the well known of the lower order methods [1,2,3,5,7,12-14]. This paper is actually a continuation form the previous study [11]. The prime motive for presentation of the new optimal-order method was to increase the eighth-order convergence method given in [14] and support the Kung and Traub conjecture [8]; the multipoint iteration methods without memory, based on \( n \) evaluations could achieve optimal convergence order \( 2^{n-1} \). Thus, we present a sixteenth-order method which agrees with Kung and Traub conjecture for \( n = 5 \). Consequently, we find that the new family of the Newton-Ostrowski methods is efficient and robust.

2. Construction of the Method

We construct a new family of four-point methods of the Newton-Ostrowski type with the optimal order sixteen using five functions evaluations. The first three steps are same as those of the eighth-order method introduced in [14], while the fourth step is constructed using parametric functions which are determined in such a way that the order of convergence of four-step method is sixteen. To obtain the solution of (1) by the new methods, we must evaluate the first derivative of (1) and set a particular initial approximation \( x_0 \), ideally close to the simple root. We
begin with the first three steps of the Newton-Ostrowski eighth-order method and then followed by the fourth improved step, hence four point method with optimal order.

\[ u_n = x_n - \frac{f(x_n)}{f'(x_n)}, \]  

(2)

\[ v_n = u_n - \left( \frac{f(u_n)}{f(x_n) - 2f(u_n)} \right) \left( \frac{f(x_n)}{f'(x_n)} \right), \]  

(3)

\[ w_n = v_n - \left( \sum_{i=0}^{3} \theta_i \right) \left( \frac{f(v_n)}{f'(x_n)} \right), \]  

(4)

\[ x_{n+1} = w_n - \left( \sum_{i=0}^{p} \theta_i \right) \left( \frac{f(w_n)}{f'(x_n)} \right). \]  

(5)

Where \( n, p \in \mathbb{N} \) provided that the denominators (2) - (5) are not equal to zero and the seven real-valued functions; \( \theta_0 = 1 \).

\[ \theta_1 = \left[ \frac{f(x_n) - f(u_n)}{f(x_n) - 2f(u_n)} \right]^2 - 1. \]  

(7)

\[ \theta_2 = \left( \frac{f(v_n)}{f(u_n) - 2f(v_n)} \right). \]  

(8)

\[ \theta_3 = \frac{4f(v_n)}{f(x_n)}. \]  

(9)

\[ \theta_4 = \left( \frac{f(w_n)}{f(v_n)} \right) + \left( \frac{f(v_n)}{f(x_n)} \right) \left( \frac{f(u_n)}{f'(x_n)} \right). \]  

(10)

\[ \theta_5 = 2 \left( \frac{f(w_n)}{f(v_n)} \right) \left( \frac{f(u_n)}{f(x_n)} \right) + 4 \left( \frac{f(u_n)}{f'(x_n)} \right)^2 \left( \frac{f(v_n)}{f(x_n)} \right) + 2 \left( \frac{f(v_n)}{f(u_n)} \right) \left( \frac{f(v_n)}{f(x_n)} \right). \]  

(11)

\[ \theta_6 = 2 \left( \frac{f(w_n)}{f(u_n)} \right) + 13 \left( \frac{f(v_n)}{f(x_n)} \right) \left( \frac{f(u_n)}{f(x_n)} \right)^3 - 3 \left( \frac{f(v_n)}{f(x_n)} \right)^2 + 5 \left( \frac{f(w_n)}{f(v_n)} \right) \left( \frac{f(u_n)}{f(x_n)} \right)^2. \]  

(12)

\[ \theta_7 = 8 \left( \frac{f(w_n)}{f(x_n)} \right) + 12 \left( \frac{f(w_n)}{f(v_n)} \right) \left( \frac{f(u_n)}{f(x_n)} \right)^3 - 2 \left( \frac{f(v_n)}{f(u_n)} \right)^2 \left( \frac{f(v_n)}{f(x_n)} \right) \] 

\[ - 22 \left( \frac{f(v_n)}{f(x_n)} \right)^2 \left( \frac{f(u_n)}{f(x_n)} \right) + 38 \left( \frac{f(v_n)}{f(x_n)} \right) \left( \frac{f(u_n)}{f(x_n)} \right)^4. \]  

(13)
It is well established that (2), (3), (4) have an order of convergence 2, 4, 8, respectively. For the purpose of this paper we construct new methods which have the order of convergence from 9 to 16. Thus, the value of \( p \) in (5) determines the order of convergence of the new Newton-Ostrowski method. We display the different order of convergence produced by (5) depending on \( p \) in the tables. Consequently, the Newton-Ostrowski method given by (5) produces an optimal order of convergence when \( p = 7 \) and \( \lambda = 1 \). Furthermore, the real-valued function \( \tilde{\Theta}_1 \) given in (9) can take many forms; hence we state some of them below as

\[
\tilde{\Theta}_1 = \left[ 1 - 2 \frac{f(u_n)}{f(x_n)} + \left( \frac{f(u_n)}{f(x_n)} \right)^2 \right]^{-1}.
\]

(14)

\[
\tilde{\Theta}_1 = \left[ 1 - 2 \left( \frac{f(u_n)}{f(x_n)} \right) - \left( \frac{f(u_n)}{f(x_n)} \right)^2 + \left( \frac{f(u_n)}{f(x_n)} \right)^3 + 2 \left( \frac{f(u_n)}{f(x_n)} \right)^4 \right]^{-1}.
\]

(15)

\[
\tilde{\Theta}_1 = 2 \left( \frac{f(u_n)}{f(x_n)} \right) + 5 \left( \frac{f(u_n)}{f(x_n)} \right)^2 + 12 \left( \frac{f(u_n)}{f(x_n)} \right)^3 + 28 \left( \frac{f(u_n)}{f(x_n)} \right)^4 + 64 \left( \frac{f(u_n)}{f(x_n)} \right)^5.
\]

(16)

In practice, it is reasonable to choose \( \tilde{\Theta}_1 \) as simple as possible.

3. Convergence Analyses

In this section we establish the order of convergence of the new method. First we state the three essential definitions used in our examination of the new method.

**Definition 1.** Let \( f(x) \) be a real function with a simple root \( \alpha \) and let \( \{x_n\} \) be a sequence of real numbers that converge towards \( \alpha \). The order of convergence \( m \) is given by

\[
\lim_{n \to \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^m} = \rho \neq 0
\]

(17)

where \( \rho \) is the asymptotic error constant and \( m \in \mathbb{R}^+ \)

**Definition 2.** Let \( \lambda \) be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [4,10] and defined as

\[
\mu^{1/\lambda}
\]

(18)

where \( \mu \) is the order of the method.

**Definition 3** Suppose that \( x_{n-1}, x_n \) and \( x_{n+1} \) are three successive iterations closer to the root \( \alpha \) of (1). Then, the computational order of convergence [16], may be approximated by
\[
\tilde{m} \approx \frac{\ln\left|\frac{x_{n+1} - \alpha}{x_n - \alpha}\right|}{\ln\left|\frac{x_n - \alpha}{x_{n-1} - \alpha}\right|}.
\]

(19)

where \( n \in \mathbb{N} \)

In numerical mathematics it is very useful and essential to know the behaviour of an approximate method. Therefore, we prove the order of convergence of the new sixteenth-order method. Without loss of generality, we prove the new method of optimal order (sixteen) based on \( p = 7, \lambda = 1 \) and a particular real-valued function \( \theta \).

**Theorem 1**

Assume that the function \( f: I \subseteq \mathbb{R} \to \mathbb{R} \) for an open interval \( I \) have a simple root \( \alpha \in I \). Let \( f(x) \) be sufficiently smooth in the interval \( I \) and the initial approximation \( x_0 \) is sufficiently close to \( \alpha \), then the optimal order of convergence of the new method defined by (5) is sixteen when \( \lambda = 1 \).

**Proof**

Let \( \alpha \) be a simple root of \( f(x) \), i.e. \( f(\alpha) = 0 \) and \( f'(\alpha) \neq 0 \), and the error is expressed as \( e = x - \alpha \).

Using Taylor expansion, we have

\[
f(x_n) = f(\alpha) + f'(\alpha)e_n + 2^{\frac{n}{2}}f''(\alpha)e_n^2 + 6^{\frac{n}{3}}f'''(\alpha)e_n^3 + 24^{\frac{n}{4}}f''''(\alpha)e_n^4 + \cdots.
\]

(20)

Taking \( f(\alpha) = 0 \) and simplifying, expression (20) becomes

\[
f(x_n) = f'(\alpha)e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots.
\]

(21)

where \( n \in \mathbb{N} \) and

\[
c_k = \frac{f^{(k)}(\alpha)}{(k!)f'(\alpha)} \quad \text{for} \quad k = 2, 3, 4, \ldots
\]

(22)

Furthermore, we have

\[
f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots\right].
\]

(23)

Dividing (21) by (23), we get

\[
\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2\left(2c_2 - c_3\right)e_n^3 + \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + \cdots.
\]

(24)

and hence, we have

\[
d_n = e_n - \frac{f(x_n)}{f'(x_n)} = c_2e_n^2 - 2\left(2c_2 - c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + \cdots.
\]

(25)

The expansion of \( f(u_n) \) about \( \alpha \) is given as

\[
f(u_n) = f'(\alpha)\left[d_n + c_2d_n^2 + c_3d_n^3 + c_4d_n^4 + \cdots\right].
\]

(26)

Substitute (25) into (26), we obtain

\[
f(u_n) = f'(\alpha)\left[c_2e_n^2 - 2\left(2c_2 - c_3\right)e_n^3 - \left(7c_2c_3 - 5c_2^3 - 3c_4\right)e_n^4 + \cdots\right].
\]

(27)
Since from (3) we have
\[ v_n - \alpha = u_n - \alpha - \left( \frac{f'(u_n)}{f(x_n) - 2f(u_n)} \right) = c_2 \left( c_2^2 - c_3 \right)e_n^4 + \cdots, \] (28)

Taylor expansion of \( f(v_n) \) about \( \alpha \) is
\[ f(v_n) = f'(\alpha) \left[ (v_n - \alpha) + c_2 (v_n - \alpha)^2 + c_3 (v_n - \alpha)^3 + \cdots \right]. \] (29)

In order to evaluate the essential terms of (4), we expand term by term
\[ \theta_0 = 1. \] (30)
\[ \theta_1 = 2c_2e_n - (c_2^2 - 4c_4)e_n^2 - 2(c_2^3 - 3c_4)e_n^3 + \cdots, \] (31)
\[ \theta_2 = (c_2^2 - c_3)e_n^2 + (4c_2c_3 - 2c_3^2 - 2c_4)e_n^3 + \cdots, \] (32)
\[ \theta_3 = (4c_2^3 - 4c_2c_4)e_n^3 + (36c_2^2c_3 - 8c_3^2 - 20c_4^2 - 8c_4c_4)e_n^4 + \cdots. \] (33)

Since from (4) we obtain
\[ w_n = v_n - \alpha - \left[ \theta_0 + \theta_1 + \theta_2 + \theta_3 \right] \left( \frac{f(v_n)}{f'(x_n)} \right). \] (34)

Taylor expansion of \( f(w_n) \) about \( \alpha \) is
\[ f(w_n) = f'(\alpha) \left[ (w_n - \alpha) + c_2 (w_n - \alpha)^2 + c_3 (w_n - \alpha)^3 + \cdots \right]. \] (35)

Further expansion of terms used in (5) are given as
\[ \theta_4 = (13c_2^4 - 14c_2^2c_3 + c_2c_4)e_n^4 - (86c_2^5 - 160c_3^2c_3 + 30c_2^2c_4 - 2c_3c_4 + 48c_2c_3^2 - 2c_2c_4)e_n^5 + \cdots, \] (36)
\[ \theta_5 = (30c_2^5 + c_2^2c_4 - 34c_2c_3 - 2c_3c_4)e_n^5 + \cdots, \] (37)
\[ \theta_6 = (94c_2^6 - 2c_3c_4 + 7c_3^2c_4 + 23c_2^2c_3^2 - 122c_2c_4)e_n^6 + \cdots, \] (38)
\[ \theta_7 = (255c_2^7 + c_2^2c_4 + 79c_3c_3^2 - 347c_2^5c_3 - 8c_2c_3c_4 + 20c_3c_4)e_n^7 + \cdots. \] (39)

Since from (5) we obtain
\[ e_{n+1} = w_n - \alpha - \left[ \sum_{i=0}^{2} \theta_i \left( \frac{f(w_n)}{f'(x_n)} \right) \right]. \] (40)

Substituting appropriate expressions in (5), we obtain the error equation
\[ e_{n+1} = \left[ 8052c_2^{15} - c_2^6c_3^2e_4^2 + 2c_2c_4c_4c_6^2 - 12c_2c_2c_6^2 + 37c_2c_4c_2^8 - 38c_2c_3c_2^7 + 13c_2c_3^3c_2^5 + 1499c_2c_4c_2^2 \\
- c_2c_4c_4 - 139c_2^3c_4c_2^2 - 2c_2^3c_2c_4 + 30c_2^2c_2c_4 - 3922c_4c_2c_2c_2^2 - 3338c_2c_3c_2c_2^2 + 44c_2c_4c_2c_4^2 - 959c_2c_4c_2c_4^2 \\
+ 81c_2c_4^2 + c_2c_4^2 - 28607c_4c_2c_2^2 + 37045c_2^3c_2^2 - 20684c_2c_2^4 + 4428c_2c_2^7 - 234c_2^3c_2^5 \right] e_n^{16}. \] (41)

The expression (41) establishes the asymptotic error constant for the sixteenth order of convergence for the Newton-Ostrowski method defined by (5).
Furthermore, it is elementary to prove the order of convergence of the other new methods; therefore, we state the asymptotic error constant (AEC(m)) for the other lower order Newton-Ostrowski methods.

\[
AEC(15) = \left[ 3048 c_2^{14} + 8c_2^4 c_1^2 c_4 - 2c_2^4 c_3^2 c_4 - 178c_2^5 c_3 c_4 + 880c_2^7 c_3^2 c_4 - 1194c_2^9 c_3 c_4 - 28c_2^6 c_3^2 c_4^2 - 10478c_2^{12} c_3 + 128140c_2^{10} c_3^3 - 6348c_2^8 c_3^4 + 26c_2^4 c_3^5 + 938c_2^6 c_3^4 + 20c_2^8 c_3^2 + 49c_2^{11} c_4 \right] e_n^{15}.
\]

(42)

\[
AEC(14) = \left[ 1128 c_2^{11} + 2c_2^4 c_3^2 c_4 - 49c_2^4 c_3^3 c_4 + 286c_2^6 c_3^2 c_4 - 415c_2^8 c_3 c_4 - 9c_2^5 c_3 c_4^2 - 3814c_2^{11} c_3 + 4548c_2^9 c_3^3 - 2161c_2^7 c_3^2 + 299c_2^5 c_3^4 + 7c_2^7 c_3^2 + 178c_2^{10} c_4 \right] e_n^{14}.
\]

(43)

\[
AEC(13) = \left[ 360c_2^{12} - 1158c_2^{10} c_3 + 54c_2^9 c_4 - 114c_2^7 c_3 c_4 + 2c_2^6 c_4^2 - 492c_2^5 c_3^3 + 26c_2^4 c_3^4 - 2c_2^3 c_3^5 c_4 + 1264c_2^6 c_3^2 - 62c_2^5 c_3^4 - 2c_2^4 c_3^2 c_4^2 \right] e_n^{15}.
\]

(44)

\[
AEC(12) = \left[ 156c_2^{11} + 25c_2^8 c_4 - c_2^4 c_3 c_4^2 + c_2^4 c_3^2 c_4 - 27c_2^5 c_3^2 c_4 - 52c_2^5 c_3 c_4 - 493c_2^9 c_4 + 519c_2^7 c_3^3 - 18c_2^5 c_3^2 \right] e_n^{12}.
\]

(45)

\[
AEC(11) = \left[ 48c_2^{10} + 4c_2^7 c_4 + 4c_2^5 c_3 c_4 - 8c_2^6 c_3 c_4 - 148c_2^8 c_3 - 52c_2^4 c_3^3 + 152c_2^6 c_3^2 \right] e_n^{11}.
\]

(46)

\[
AEC(10) = \left[ 12c_2^9 + c_2^6 c_4 + c_2^5 c_3^2 c_4 - 13c_2^4 c_3^3 - 37c_2^4 c_3 c_4 + 38c_2^5 c_3^2 \right] e_n^{10}.
\]

(47)

\[
AEC(9) = \left[ 24c_2^8 + 2c_2^5 c_4 - 2c_2^4 c_3 c_4 + 26c_2^4 c_3^2 - 50c_2^6 c_3 \right] e_n^9.
\]

(48)

\[
AEC(8) = \left[ 12c_2^7 + c_2^4 c_4 - c_2^3 c_3 c_4 + 13c_2^5 c_3^2 - 25c_2^6 c_3 \right] e_n^8.
\]

(49)

The asymptotic error constant (49)-(42) represents the order of convergence of the Newton-Ostrowski method form 8 to 15, respectively. The new Newton-Ostrowski method requires a total of five evaluations of the function. Using (18), we determine the efficiency index of the new sixteenth-order methods. Therefore, the new optimal Newton-Ostrowski sixteenth-order method given by (5) has the efficiency index of \( \sqrt[16]{16} \approx 1.741 \) which is better than the other lower order methods, for example four point methods of lower order are given as \( \sqrt[4]{4} \approx 1.695 \), \( \sqrt[3]{3} \approx 1.670 \), \( \sqrt[2]{2} \approx 1.644 \), \( \sqrt[10]{10} \approx 1.615 \), \( \sqrt[5]{5} \approx 1.585 \), \( \sqrt[8]{8} \approx 1.682 \), the efficiency index of the three point methods having eight-order convergence is given as \( \sqrt[8]{8} \approx 1.682 \), whereas the efficiency index of the two point methods is \( \sqrt[4]{4} \approx 1.695 \). We can see that the efficiency index of the new optimal Newton-Ostrowski of sixteenth-order is better than the other similar methods.

4. The Established Methods

For the purpose of comparison, we consider three higher-order methods presented recently in [5,9,13]. Since these methods are well established, we state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new sixteenth order methods.

4.1 The Geum and Kim Method

In [5], Geum et al. developed a biparametric family of optimally convergent sixteenth-order multipoint methods with their fourth-step weighting function as a sum of a rational and generic two-variable function. Since this method is well known we state the essentials expressions used in our calculation,
\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ z_n = y_n - K_f(u_n) \frac{f(y_n)}{f'(x_n)} \]
\[ s_n = z_n - H_f(u_n, v_n, w_n) \frac{f(z_n)}{f'(x_n)} \]
\[ x_{n+1} = s_n - W_f(u_n, v_n, w_n, t_n) \frac{f(s_n)}{f'(x_n)} \]

where

\[ K_f(u_n) = \frac{1 + \beta u_n + \left( -9 + \frac{\beta}{2} \right) u_n^2}{1 + (\beta - 2) u_n + \left( -4 + \frac{\beta}{2} \right) u_n^2} \]
\[ H_f(u_n, v_n, w_n) = \frac{1 + 2 u_n + (2 + \sigma) w_n}{1 - v_n + \sigma w_n} \]
\[ W_f(u_n, v_n, w_n, t_n) = \frac{1 + 2 u_n + (2 + \sigma) v_n w_n}{1 - v_n - 2 w_n - t_n + 2 (1 + \sigma) v_n w_n} + G(u_n, w_n) \]
\[ u_n = \frac{f(y_n)}{f(x_n)}, \quad v_n = \frac{f(z_n)}{f(y_n)}, \quad w_n = \frac{f(z_n)}{f(x_n)}, \quad t_n = \frac{f(s_n)}{f(z_n)} \]

There are many versions of \( G(u_n, w_n) \), see [5], for the purpose of this paper we shall consider the following

\[ G(u_n, w_n) = -\frac{1}{2} \left[ u_n w_n \left( 6 + 12 u_n + u_n^2 \left( 24 - 11 \beta \right) + u_n^2 \phi_n^0 + 4 \sigma \right) \right] + \phi_n^2 w_n^2 \]
\[ \phi_1^0 = 11 \beta^2 - 66 \beta + 136, \quad \phi_2^0 = 2 u_n \left( \sigma^2 + 9 \right) - 4 \sigma - 6, \quad \beta = 2, \quad \sigma = -2 \]

### 4.2 The Neta Method

In [9], Neta developed a four-step fourteen order of convergence method, since this method is well known we state the essential expressions used in the method.

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ z_n = y_n - L(u_n) \frac{f(y_n)}{f'(x_n)} \]
\[ s_n = z_n - P(u_n) \frac{f(z_n)}{f'(x_n)} \]
\[ x_{n+1} = y_n + \left( \sum_{i=1}^{3} \theta_i f^i(x_n) \right) \]

where

\[ u_n = \frac{f(y_n)}{f(x_n)}, \quad L(u_n) = \frac{1 + \beta u_n}{1 + (\beta - 2) u_n}, \quad P(u_n) = \frac{1 - u_n}{1 - 3 u_n} \]
\theta_3 = \frac{\Delta_1 - \Delta_2}{F_u - F_x}, \quad \theta_2 = -\Delta_1 + \theta_1 (F_u - F_x), \quad \theta_1 = \varphi_u + \theta_2 F_u - \theta_1 F_u^2, \quad (58)

\Delta_1 = \frac{\varphi_u - \varphi_x}{F_u - F_x}, \quad \Delta_2 = \frac{\varphi_y - \varphi_z}{F_y - F_z}, \quad (59)

\varphi_u = \frac{1}{F_u} \left( \frac{w_n - z_n}{F_u} - \frac{1}{f'(x_n)} \right), \quad (60)

\varphi_y = \frac{1}{F_y} \left( \frac{y_n - z_n}{F_y} - \frac{1}{f'(x_n)} \right), \quad (61)

\varphi_z = \frac{1}{F_z} \left( \frac{z_n - x_n}{F_z} - \frac{1}{f'(x_n)} \right), \quad (62)

F_u = f(w_n) - f(x_n), \quad F_y = f(y_n) - f(x_n), \quad F_z = f(z_n) - f(x_n), \quad (63)

For further details of the above method maybe found in [9,13].

4.3 The Soleymani Method

In [13], Soleymani developed a general efficient class of four-step fourteen order root finding method. As before, we state the essential expression used in the calculation of the method,

\[
\begin{align*}
    y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
z_n &= y_n - L \frac{f(y_n)}{f'(x_n)} \\
s_n &= z_n - Q \frac{f(z_n)}{f'(x_n)} \\
x_{n+1} &= s_n - M f(s_n)
\end{align*}
\]

where

\[
u_n = \frac{f(y_n)}{f(x_n)}, \quad L = \frac{1 + \beta u_n}{1 + (\beta - 2) u_n}, \quad (65)
\]

\[
Q = \frac{(1 + a_4(z_n - x_n))^2}{f'(x_n) + a_3(z_n - x_n)(2 + a_4(z_n - x_n))}, \quad (66)
\]

\[
M = \frac{(1 + b_4(w_n - x_n))^2}{f'(x_n) + b_3(z_n - x_n)(2 + b_4(z_n - x_n))}, \quad (67)
\]
Consequently, we observe that the rating the effectiveness of the new optimal order iterative methods and the exact solution gives estimates of the root of (72) are $\alpha = -1.16835388...$. In Table 1 are the errors obtained by each of the Newton Ostrowski type methods described, based on the initial approximation $x_0 = -2$. We observe that the new optimal order Newton-Ostrowski iterative method is converging of the order sixteen.

5. Application of the New Optimal Order Iterative Methods

To demonstrate the performance of the new Newton-Ostrowski type methods, we take four particular nonlinear equations. Thus, we determine the consistency and stability of results by examining the convergence of the new iterative method. The findings are generalised by illustrating the effectiveness of the Newton-Ostrowski type methods for determining the simple root of a nonlinear equation. Consequently, we give estimates of the approximate solution produced by the Newton-Ostrowski type methods and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In addition, the errors displayed are of absolute value.

5.1 Numerical Example 1

In our first example we demonstrate the convergence of the new Newton-Ostrowski iterative methods for the following nonlinear equation

$$f(x) = x^{13} - x - 10,$$

and the exact value of the simple root of (72) is $\alpha = -1.16835388...$. In Table 1 are the errors obtained by each of the Newton Ostrowski type methods described, based on the initial approximation $x_0 = -2$. We observe that the new optimal order Newton-Ostrowski iterative method is converging of the order sixteen.

| methods | $|x_1 - \alpha|$ | $|x_2 - \alpha|$ | $|x_3 - \alpha|$ | $|x_4 - \alpha|$ | $m$ |
|---------|----------------|----------------|----------------|----------------|-----|
| (5)     | 0.515          | 0.253          | 0.592e-1       | 0.575e-3       | 3.20 |
| (6), $z=1$ | 0.444          | 0.142          | 0.231e-2       | 0.191e-15      | 7.32 |
| (7), $p=0$ | 0.437          | 0.132          | 0.118e-2       | 0.116e-19      | 8.30 |
| (7), $p=1$ | 0.410          | 0.955e-1       | 0.376e-4       | 0.159e-37      | 9.80 |
| (7), $p=2$ | 0.408          | 0.925e-1       | 0.231e-4       | 0.635e-43      | 10.71 |
| (7), $p=3$ | 0.405          | 0.894e-1       | 0.129e-4       | 0.219e-49      | 11.66 |
| (7), $p=4$ | 0.401          | 0.844e-1       | 0.489e-5       | 0.115e-58      | 12.66 |
| (7), $p=5$ | 0.398          | 0.803e-1       | 0.203e-5       | 0.405e-68      | 13.64 |
| (7), $p=6$ | 0.393          | 0.750e-1       | 0.600e-6       | 0.132e-80      | 14.65 |
| (7), $p=7$ | 0.389          | 0.697e-1       | 0.149e-6       | 0.224e-95      | 15.67 |
| (56)    | 0.461          | 0.168          | 0.548e-2       | 0.570e-20      | 6.57 |
| (64)    | 0.309          | 0.102e-1       | 0.503e-18      | 0.445e-246     | 13.89 |
| (50)    | 0.444          | 0.157          | 0.173e-3       | 0.370e-47      | 12.17 |
5.2 Numerical Example 2

In our second example we demonstrate the convergence of new Newton-Ostrowski iterative methods for a different type of nonlinear equation

\[f(x) = \sin(x)e^{-x} + \ln(1 + x^2),\]

and the exact value of the simple root of (73) is \(\alpha = 0\). In Table 2 are the errors obtained by each of the methods described, based on the initial approximation \(x_0 = 1\). In this particular case we observe that all the Newton-Ostrowski type methods are performing better than expected.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Errors occurring in the estimates of the root of (73) by the methods described</th>
</tr>
</thead>
<tbody>
<tr>
<td>methods</td>
<td>(</td>
</tr>
<tr>
<td>(5)</td>
<td>0.126e-1</td>
</tr>
<tr>
<td>(6), (\lambda=1)</td>
<td>0.342e-3</td>
</tr>
<tr>
<td>(7), (p=0)</td>
<td>0.426e-4</td>
</tr>
<tr>
<td>(7), (p=1)</td>
<td>0.318e-4</td>
</tr>
<tr>
<td>(7), (p=2)</td>
<td>0.101e-4</td>
</tr>
<tr>
<td>(7), (p=3)</td>
<td>0.925e-5</td>
</tr>
<tr>
<td>(7), (p=4)</td>
<td>0.175e-5</td>
</tr>
<tr>
<td>(7), (p=5)</td>
<td>0.218e-5</td>
</tr>
<tr>
<td>(7), (p=6)</td>
<td>0.665e-6</td>
</tr>
<tr>
<td>(7), (p=7)</td>
<td>0.427e-6</td>
</tr>
<tr>
<td>(56)</td>
<td>0.873e-6</td>
</tr>
<tr>
<td>(64)</td>
<td>0.100e-7</td>
</tr>
<tr>
<td>(50)</td>
<td>0.395e-6</td>
</tr>
</tbody>
</table>

5.3 Numerical Example 3

In this subsection we take another nonlinear equation. We demonstrate the convergence of the new Newton-Ostrowski iterative methods for the following nonlinear equation

\[f(x) = xe^{x^2} - \sin(x)^2 + 3\cos(x) + 5,\]

and the exact value of the simple root of (74) is \(\alpha = -1.207648...\). In Table 3 are the errors obtained by each of the methods described, based on the initial approximation \(x_0 = -2\). Here, we observe that all the Newton-Ostrowski type methods are converging of the expected order.

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Errors occurring in the estimates of the root of (74) by the methods described</th>
</tr>
</thead>
<tbody>
<tr>
<td>Methods</td>
<td>(</td>
</tr>
<tr>
<td>(5)</td>
<td>0.258</td>
</tr>
<tr>
<td>(6), (\lambda=1)</td>
<td>0.146</td>
</tr>
<tr>
<td>(7), (p=0)</td>
<td>0.138</td>
</tr>
<tr>
<td>(7), (p=1)</td>
<td>0.991e-1</td>
</tr>
<tr>
<td>(7), (p=2)</td>
<td>0.971e-1</td>
</tr>
<tr>
<td>(7), (p=3)</td>
<td>0.947e-1</td>
</tr>
<tr>
<td>(7), (p=4)</td>
<td>0.910e-1</td>
</tr>
<tr>
<td>(7), (p=5)</td>
<td>0.878e-1</td>
</tr>
<tr>
<td>(7), (p=6)</td>
<td>0.837e-1</td>
</tr>
<tr>
<td>(7), (p=7)</td>
<td>0.793e-1</td>
</tr>
<tr>
<td>(56)</td>
<td>0.171</td>
</tr>
<tr>
<td>(64)</td>
<td>0.282e-1</td>
</tr>
<tr>
<td>(50)</td>
<td>0.159</td>
</tr>
</tbody>
</table>
5.4 Numerical Example 4

In the last but not least of the examples, we take another different type of nonlinear equation. We demonstrate the convergence of new Newton-Ostrowski iterative methods for the following nonlinear equation

\[ f(x) = \tan(x) + \cos(x) + \sin(x) - 1. \quad (75) \]

and the exact value of the simple root of (75) is \( \alpha = 0 \). In Table 4 are the errors obtained by each of the methods described, based on the initial approximation \( x_0 = 2^{-1} \). Here also, we find that all the Newton-Ostrowski type methods are converging of the expected order.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( x_1 - \alpha )</th>
<th>( x_2 - \alpha )</th>
<th>( x_3 - \alpha )</th>
<th>( \bar{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5) ( \lambda = 1 )</td>
<td>0.124 e-2</td>
<td>0.123e-13</td>
<td>0.120e-57</td>
<td>4.00</td>
</tr>
<tr>
<td>(5), ( \lambda = 0 )</td>
<td>0.504e-5</td>
<td>0.566e-46</td>
<td>0.143e-373</td>
<td>8.00</td>
</tr>
<tr>
<td>(7), ( p = 0 )</td>
<td>0.902e-6</td>
<td>0.267e-58</td>
<td>0.467e-531</td>
<td>9.00</td>
</tr>
<tr>
<td>(7), ( p = 1 )</td>
<td>0.174e-6</td>
<td>0.712e-73</td>
<td>0.948e-737</td>
<td>10.00</td>
</tr>
<tr>
<td>(7), ( p = 2 )</td>
<td>0.431e-7</td>
<td>0.268e-86</td>
<td>0.146e-957</td>
<td>11.00</td>
</tr>
<tr>
<td>(7), ( p = 3 )</td>
<td>0.221e-7</td>
<td>0.502e-97</td>
<td>0.947e-1173</td>
<td>12.00</td>
</tr>
<tr>
<td>(7), ( p = 4 )</td>
<td>0.328e-8</td>
<td>0.983e-116</td>
<td>0.154e-1513</td>
<td>13.00</td>
</tr>
<tr>
<td>(7), ( p = 5 )</td>
<td>0.673e-9</td>
<td>0.436e-134</td>
<td>0.985e-1887</td>
<td>14.00</td>
</tr>
<tr>
<td>(7), ( p = 6 )</td>
<td>0.312e-9</td>
<td>0.166e-148</td>
<td>0.127e-2237</td>
<td>15.00</td>
</tr>
<tr>
<td>(7), ( p = 7 )</td>
<td>0.672e-10</td>
<td>0.544e-169</td>
<td>0.185e-2714</td>
<td>16.00</td>
</tr>
<tr>
<td>(56)</td>
<td>0.190e-10</td>
<td>0.930e-155</td>
<td>0.405e-2175</td>
<td>14.00</td>
</tr>
<tr>
<td>(64)</td>
<td>0.228e-9</td>
<td>0.192e-141</td>
<td>0.178e-1990</td>
<td>14.00</td>
</tr>
<tr>
<td>(50)</td>
<td>0.273e-12</td>
<td>0.197e-206</td>
<td>0.107e-3312</td>
<td>16.00</td>
</tr>
</tbody>
</table>

6. Remarks and Conclusion

In this paper, we have demonstrated the performance of the new nine to sixteen order iterative methods, namely the Newton-Ostrowski type methods. The prime motive of presenting these new methods was to establish a higher order of convergence method than the existing four to eight order methods [1-3,6,7,12,14]. We have examined the effectiveness of the new methods by showing the accuracy of the simple root of a nonlinear equation. After an extensive experimentation we were not able to designate a specific iterative method which always produces the best results for all tested nonlinear equations. The main purpose of demonstrating the new Newton-Ostrowski type methods for four types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. We have shown numerically and verified that the new Newton-Ostrowski methods converge of the order nine to fifteen and the optimal order sixteen is achieved when \( p = 7 \) and \( \lambda = 1 \) in (7). Furthermore, we find that the family of the new Newton-Ostrowski method only exists for the order of convergence of 9 to 14. Hence, we conjecture that further investigation is needed to establish a family for the fifteenth and the optimal sixteenth order of convergence methods. In addition, it should be noted that like all other iterative methods, the new method has its own domain of validity and in certain circumstances should not be used. Finally, we conclude that the new four-point methods may be considered a very good alternative to the classical methods.

Acknowledgement

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References


