Thermal Stresses of a Thin Annular Disc by Using Integral Transform

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Abstract: In this paper, an attempt has been made to determine the temperature distribution, displacement and thermal stresses of a thin annular disc occupying the space $D: a \leq r \leq b, -h \leq z \leq h$, with boundary conditions of radiation type. We apply integral transform techniques to find the thermoelastic solution.

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1. Introduction

Nowacki [1] has determined steady-state thermal stresses in a thick circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Roy Choudhuri [2] has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of circular upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Wankhede [3] has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature.

In all aforementioned investigations they have not considered any thermoelastic problems with boundary conditions of radiation type.

This paper is concerned with transient thermoelastic problem of a thin annular disc occupying the space $D: a \leq r \leq b, -h \leq z \leq h$, with boundary conditions of radiation type.

2. Statement of the Problem

Consider a thin annular disc of thickness $2h$ occupying the space $D: a \leq r \leq b, -h \leq z \leq h$, the material is homogeneous and isotropic. The differential equation governing to the displacement function $U(r, z, t)$ is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} = (1 + \nu) a T$$

(1)

$$U_r = 0 \quad \text{at} \quad r = a, b$$

(2)

where $\nu$ and $a_T$ are Poisson’s ratio and the linear coefficient of thermal expansion of the material of the plate and $T$ is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k} \frac{\partial T}{\partial t}$$

(3)

Subject to initial condition

$$M_r(T, 1, 0, 0) = \theta(r, z) \quad \text{for all} \quad a \leq r \leq b, -h \leq z \leq h$$

(4)

The boundary conditions are

$$M_r(T, 1, k_1, a) = F_1(z, t) \quad \text{for all} \quad -h \leq z \leq h, t > 0$$

(5)
The most general expression for these conditions can be given by

\[ M_v(f, k, \bar{k}, \bar{s}) = (\bar{k}f + k\bar{f})_{v=s} \]

where the prime (') denotes differentiation with respect to \( v \).

The stress function \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) are given by

\[ \sigma_{rr} = -2\mu \frac{1}{r} \frac{\partial U}{\partial r} \]
\[ \sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \]

where \( \mu \) is the Lame’s constant, while each of the stress functions \( \sigma_{rr}, \sigma_{\theta\theta} \) and \( \sigma_{k\kappa} \) are zero within the plate in the plane state of stress.

The equations (1) to (10) constitute the mathematical formulation of the problem under consideration.

3. Solution of the Problem

3.1 Transient Heat Conduction Analysis

In order to solve equation (2) under the boundary condition (4) we first introduce the method of Marchi-Fasulo transform and Marchi-Zgrablich transform of order \( n \) over the variable \( r \). Let \( n \) be the parameter of the transform, then the integral transform and its inversion theorems are written

\[ \int_a^b g(r, z, t) \frac{g^*(\xi_n, z, t)}{\mu^*} d\xi_n = \sum_{n=1}^{\infty} g^*(\xi_n, z, t) S_0(k_1, k_2, \mu_n r) \]

And

\[ f^*(r, \lambda_n, t) = \int_{-h}^{h} f(r, z, t) P_n(z) dz = \sum_{n=1}^{\infty} \frac{f^*(r, \lambda_n, t) P_n(z)}{\lambda_n} \]

Applying the transform defined in equation (11) to the equation (3), one obtains

\[ k \left[ \frac{d^2\overline{T}(m, z, t)}{dz^2} - \mu_m^2 \overline{T}(m, z, t) + \Psi \right] = \frac{d\overline{T}(m, z, t)}{dt} \]

where \( \Psi = \frac{b}{\beta_2} S_0(k_1, k_2, \mu_m b) F_3(z, t) - \frac{a}{\alpha_2} S_0(k_1, k_2, \mu_m a) F_1(z, t) \)

Applying the transform defined in equation (12) to the equation (13) we obtain,

\[ k \left[ -p^2 \overline{T}(m, z, t) + \Psi + \Phi \right] = \frac{d\overline{T}(m, z, t)}{dt} \]

where \( p^2 = (\mu_m^2 + \lambda_n^2) \) and \( \Phi = \frac{P_n(h)}{k_3} F_3(r, t) - \frac{P_n(-h)}{k_4} F_4(r, t) \).
Equation (14) is a first order differential equation whose solution is given by

\[ T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n^2} \left[ \Omega + \tau e^{-\rho^2 kr} \right] \frac{S_0(k_1, k_2, \mu_m r)}{\mu_m^2} \]  

(15)

where \( \Omega = \frac{\Psi + \Phi}{\rho^2} \), \( \tau = \bar{\omega}(m, n) - \Omega \)

\[ P_n(z) = Q_n \cos(\eta_n z) - W_n \sin(\eta_n z), \]

\[ Q_n = a_n (\alpha_1 + \alpha_2) \cos(\eta_n h) + (\beta_1 - \beta_2) \sin(\eta_n h), \]

\[ W_n = (\beta_1 + \beta_2) \cos(\eta_n h) + (\alpha_2 - \alpha_1) a_n \sin(\eta_n h), \]

\[ \lambda_n = \int_{-h}^{h} P_n^2(z) dz = h \left[ Q_n^2 + W_n^2 \right] + \frac{\sin(2a_n h)}{2a_n} \left[ Q_n^2 - W_n^2 \right] \]

The eigen values \( a_n \) are the solutions of the equation

\[ \left[ \alpha_1 a \cos(ah) + \beta_1 \sin(ah) \right] \times \left[ \beta_2 \cos(ah) + \alpha_2 a \sin(ah) \right] = \left[ \alpha_2 a \cos(ah) - \beta_2 \sin(ah) \right] \times \left[ \beta_1 \cos(ah) - \alpha_1 a \sin(ah) \right] \]

\( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are constants,

the kernel function \( S_p(\alpha, \beta, \mu_m r) \) can be defined as

\[ S_p(\alpha, \beta, \mu_m r) = J_p(\mu_m r) \left[ Y_p(\alpha, \mu_m a) + Y_p(\beta, \mu_m b) \right] - Y_p(\mu_m r) \left[ J_p(\alpha, \mu_m a) + J_p(\beta, \mu_m b) \right] \]

and \( J_p(\mu r) \) and \( Y_p(\mu r) \) are Bessel function of first and second kind respectively.

The eigen values \( \mu_m \) are the positive roots of the characteristic equation

\[ J_0(k_1, \mu a) Y_0(k_2, \mu b) - J_0(k_2, \mu b) Y_0(k_1, \mu a) = 0. \]

Equation (15) is the desired solution of the given problem.

4. Thermoelastic Displacement Function

Substituting value of temperature distribution \( T(r, z, t) \) from (15) in equation (1), one obtains the thermoelastic displacement function \( U(r, z, t) \) as

\[ U = -(1 + v) a \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n^2} \left[ \Omega + \tau e^{-\rho^2 kr} \right] \frac{S_0(k_1, k_2, \mu_m r)}{\mu_m^2} \]  

(16)

5. Determination of Stress Functions

Substituting the value of thermoelastic displacement function \( U(r, z, t) \) from equation (16) in equations (9) and (10) one obtain the stress functions,
\[
\sigma_{rr} = \left( \frac{2\mu(1+v)}{r} \right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n^2} \left[ \Omega + \tau e^{-\mu^2 k^2 t} \right] S_0^{(0)}(k_1, k_2, \mu_m r) \]
\[
\sigma_{\theta\theta} = 2\mu(1+v) a_i \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n^2} \left[ \Omega + \tau e^{-\mu^2 k^2 t} \right] S_0^{(2)}(k_1, k_2, \mu_m r)
\]

6. Special Case and Numerical Results

Take \( k=0.86, \ h=2 \text{ cm}, \ b=4 \text{ cm}, \ t=1 \text{ sec}, \ r_o=1 \text{ cm} \ \omega=1 \). Substitute this values in (20), one obtains

\[
T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n^2} \left[ \Omega + \tau e^{-0.86 \mu^2} \right] S_0(0.25, 0.25, \mu_m r)
\]

7. Conclusion

In this paper, the temperature distribution, displacement function and thermal stresses have been determined for a thin annular disc. The finite Marchi-Fasulo transform and March-Zgrablich transform techniques have been used to obtain numerical results. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

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Reference


