A Supplier Strategy to Control the Bullwhip Effect

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***Abstract*— Strategies to control the Bullwhip effect in supply chains have been the focus of substantial amount of research in the last few decades. Some strategies were based on the implementation of information sharing and collaboration with retailers. Other strategies suggested the use of the vendor-managed inventory approach. Recently, a new type of strategies suggested the supplier's control on the replenishment of orders received from a pool of retailers with replenishment quantity decisions that can decrease the effect of the bullwhip phenomenon to the supplier. However, this strategy leads to preference discriminations among the retailers. This paper proposes a similar strategy that controls the fulfillment quantities but to individual retailer's orders, independently of other retailers. This strategy is capable to conserve the expected mean of the retailer's orders while reducing their expected variance. The main contribution here is to reduce the impact of the bullwhip effect on the supplier side by controlling the fulfilled quantities to the retailers. Surprisingly, this proposed strategy eventually improves the service level on the retailer side.**

***Keywords*—** *Supply chain management, Mitigating the bullwhip effect, Supplier strategies, variance reduction, service level.*

1. Introduction

The bullwhip effect is widely known as the amplification of the variability in retailers' orders that the supplier must fulfill when compared with the relatively smaller fluctuations in the market demand. Over the last few decades, a substantial amount of research was conducted to develop creative strategies that aim to control this phenomenon and attempt to minimize the amplification effect of the variance of orders as it propagates upstream the supply chain, and thus reduce the undesirable outcomes of this phenomenon on suppliers.

A new strategy is proposed in this paper to help the supplier reduce the variability in the fulfilled orders to retailers using a technique that imitates the method of control-variates for variance reduction in simulation data output.

With this strategy, the supplier controls the amount of the fulfilled orders to retailers so that when a received order from a retailer is larger than its expected mean, the supplier fulfills it with a quantity that is partially reduced, and when such order was less than its expected mean, the supplier fulfills it with a quantity that is slightly more.

As a result, the variability in the fulfilled orders will be reduced while maintaining the expected mean of the original retailer's orders.

This strategy resembles the control-variate technique in simulation modeling; see for example [1]. This research investigates how the above strategy will reduce the variance of the fulfilled orders to a given retailer and explores the impact of this strategy on the retailer's service level.

The literature review in Section 2 scans several papers that addressed the mitigation of the bullwhip effect in supply chains.

In section 3, the research methodology is provided where a mathematical model is constructed to depict the fulfilled orders by the supplier using the new strategy in terms of the original orders of the retailer. In this section as well, the mean and the variance of the fulfilled orders are derived and compared with those of the original retailer's orders.

A simulation was also conducted to validate the theoretical derivations in the mathematical model and its results are also provided in this section as well.

Section 4 discusses the managerial implementation of the new strategy in real practice and shows that this strategy of controlled fulfilled orders improves the service level to the retailer. This section also discusses how the new fulfillment strategy impact the average inventory level on the retailer side.

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Finally, the paper closes in Section 5 with a conclusion and proposed future work.

2. Literature Review

A comprehensive review on recent publications about the mitigation of the bullwhip effect in supply chain is provided in [2], which displayed the literature chronologically and classified the related research in various categories.

It was argued in [3] that the bullwhip effect will always exist for inventory policies that use base-stock level.

Existing theoretical and empirical research on the causes of the bullwhip effect using various demand patterns was reviewed by [4] and used the ratio of the upstream demand variance of the chain with respect to the market demand variance to measure the level of the Bullwhip effect. They ended up with ratios that are consistently larger than one.

Limiting bounds are derived in [5] for both low market demand rates and large review periods adopted by the supplier wo use base-stock inventory policies. These bounds were consistently strictly larger than one and affirmed the fact that the bullwhip phenomenon can only be mitigated rather than been entirely eliminated.

Mitigating strategies of the bullwhip effect using information sharing were originally suggested in [6]. The effectiveness of information sharing under various operating conditions was discussed in [2], and it was concluded in [7] that information sharing cannot eliminate the bullwhip phenomena completely.

The strategy of vendor managed inventory (VMI) as a strategy to mitigate the bullwhip effect, and where the supplier takes full control of managing the inventory of retailers, was studied in [8]. It was shown there that the VMI strategy performs better than information sharing under realistic conditions of the market demand, but its implementation requires high degree of trust and acceptance by the retailers.

A third type of strategies to mitigate the bullwhip effect was based on trust and collaboration and was suggested in [9]. It was argued in this work; however, that the success for such approaches highly depends on the behavior of the supply chain members and requires significant cultural change and close collaboration.

The new strategy in this paper is proposed for the situations where the mitigation methods of information sharing, VMI, or the behavioral collaboration may face difficulties in their implementation either technically or administratively. This strategy can be adopted by the supplier independently of the retailer, especially when their orders shows large fluctuations.

A similar supplier-controlled strategy was proposed in [10] where the portfolio theory was used to reduce the total variance of orders from a pool of retailers while maintaining the total mean of their market demand. They have considered a simple case of two retailers and used a linear programming approach to minimize the variance of the total of orders received from those retailers, subject to maintaining the total of their expected means. That result showed that the minimum variance can be attained by fulfilling the orders from the retailer who has higher variability with less quantities than what were ordered, and to fulfill to the retailer who has orders that show smaller variability with larger fulfilled amounts.

Like [10], the aim in this paper is to minimize the variance of the total orders received but it considers one single retailer at a time. With this strategy all retailers are treated equally rather than being discriminated based on the variability of their orders.

The next section proposes a mathematical formulation of the proposed strategy in order to derive and minimize the variance of the supplier's fulfilled order to a given retailer. The expected mean of the fulfilled orders is also derived to assure that this strategy meets the original mean of the retailer's orders.

3. Methodology

In order to describe how the proposed strategy can reduce the variability of a single retailer's orders, a mathematical model is established in section 3.1 to depict the fulfillment process and is used to estimate the variance of the fulfilled orders by the supplier in terms of the original retailer's order variance. This will help derive the bullwhip effect ratio for a single retailer.

Thus, it is expected that by implementing this variance reduction for a single retailer, the supplier will be able to reduce the total variance of the aggregation of its retailers' orders.

The theoretical results in section 3.1 were validated using a simulation that apply the strategy which results are shown and discussed in section 3.2. The simulation also suggested a surprising improvement in the retailer's service level. This observation as well other implementation issues will be discussed further in section 4.

3.1 The mathematical model

Consider a two-echelon supply chain that consists of a supplier and a single retailer. Assume that the retailer is using a periodic base-stock inventory policy where the retailer will end up making random orders to the supplier.

Let X be the random variable of the retailer's orders and let $f(x), x > 0$ be its density function with expected mean $\mu = E[X]$. As described earlier in this paper, when the retailer makes a new order to the supplier, let the fulfilled quantity by the supplier be equal to $X - a(X - \mu)$, where a is a positive number.

It is clear that when the received order is larger than the mean of this retailer's orders then the fulfilled quantity will be less, and on the other hand, when the received order from the retailer is smaller than the mean, the fulfilled quantity will be larger than the order. Here a acts a control parameter.

The fulfilled order by the supplier as proposed by the strategy in this paper will be denoted by X_f and can be expressed as:

$$X_f = X - a\,(X - \mu)\left(1_{X\leq\mu} + 1_{X>\mu}\right) \qquad (1)$$

where X designates the original order of the retailer.

Note that $1_{X\leq\mu}$ is an indicator function which value is 1 when the $X \leq 1$ and 0 otherwise, while $1_{X>\mu}$ is an indicator function which value is 1 when the $X > 1$ and 0 otherwise. Here, the control parameter a will regulate the amount of the fulfilled quantity for the orders received from the retailer and its value can be decided solely by the supplier.

The way that X_f was defined in (1) guarantees that the expected mean of the fulfilled orders by the supplier is equal to the mean of the original retailer's orders. This is assured by deriving its expected value as follows:

$$\begin{aligned} E[X_f] &= E[X] - a\,E[X \cdot 1_{X\leq\mu}] + a\,\mu\,E[1_{X\leq\mu}] \\ &\quad - a\,E[X \cdot 1_{X>\mu}] + a\,\mu\,E[1_{X>\mu}] \end{aligned} \qquad (2)$$

Note that $E[1_{X\leq\mu}] = \int_0^{\mu} f(x)dx = P(X \leq \mu)$ and $E[1_{X>\mu}] = \int_{\mu}^{\infty} f(x)dx = P(X > \mu)$.

Furthermore, define the following "partial" means

$$\mu'_L = E[X \cdot 1_{X\leq\mu}] = \int_0^{\mu} x\,f(x)\,dx, \qquad (3)$$

$$\mu'_R = E[X \cdot 1_{X>\mu}] = \int_{\mu}^{+\infty} x\,f(x)\,dx, \qquad (4)$$

Using those partial means, Equation (2) can be rewritten as:

$$\begin{aligned} E[X_f] &= E[X] - a\,\mu'_L + a\,\mu\,P(X \leq \mu) - a\,\mu'_R \\ &\quad + a\,\mu\,P(X > \mu) \\ &= E[X] - a\left(\mu'_L + \mu'_R\right) \\ &\quad + a\,\mu\left(P(X \leq \mu) + P(X > \mu)\right) \end{aligned} \qquad (5)$$

It is very clear that $\mu'_L + \mu'_R = \mu$, and therefore $E[X_f] = E[X] = \mu$. This means that the supplier-controlled order fulfillment strategy under consideration will meet the expected mean of the retailer's orders in the long run.

On the other hand, the derivation of $Var(X_f)$ is mathematically involved and is provided in the Appendix. That derivation leads to the following simple expression:

$$Var(X_f) = (1 - a)^2\,Var(X). \qquad (6)$$

As (6) shows, the proposed strategy can reduce the variance of the fulfilled orders when $0 < a < 2$. It is also a fact that when $a = 0$, the fulfilled quantity will be equal to the order amount as set by the customer.

3.2 Validation of the results

To validate the results obtained in the previous sections, a simulation using MS-EXCEL was conducted for around 2000 market order instances with inter-occurrence random times that have an exponential distribution at a demand rate of 1.5 units per period.

The retailer uses a periodic review, base-stock inventory policy with a period that is equal to 25. The retailer's service level was assumed 90% service level which requires an order-up-to base level that is equal to 48 units. Zero lead time was assumed in this simulation.

The simulation spanned around 52 generated orders by the retailer to the supplier. The maximum of the simulated orders was 51 units, and the minimum was 27. Their average was 36.74 units, and their standard deviation was 6.48 units.

The supplier-controlled fulfillments for the retailer orders were generated using a control parameter a equal to 0.75. The generated fulfilled quantities in this simulation had a mean of 36.44 units, practically identical to the original orders mean, and their standard deviation significantly dropped to 1.39 units.

The maximum inventory level obtained for the retailer with the fulfilled quantities was 58 and its minimum was 1 unit. This means that the service level of the retailer in this simulation was 100%.

Figure 1 shows the time signal of the inventory level that was obtained in this simulation for both the original order quantities and their supplier fulfillments.

It is worthy to notice in this simulation that the percentage increase in the average inventory level was 1.7%, which is a moderate increase in what is concerning the inventory holding costs.

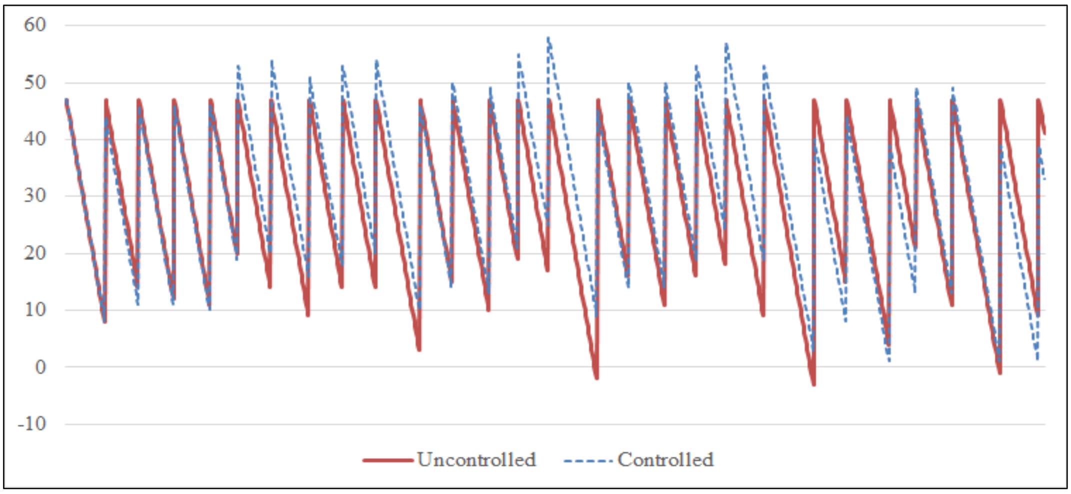

Figure 1 Retailer's inventory levels with and without supplier-controlled fulfillment orders

4. Discussions

The implementation of the proposed strategy requires special attention on how to choose the control parameter a, and as the simulation showed, this strategy will have positive impact on the retailer's service level. These are discussed in the following

4.1 Implementation issues of the new strategy

Using (6), when the control parameter a is set at 1, the variance of the fulfilled quantity can be obviously eliminated totally and end with $Var(X_f) = 0$. This, however, leads to a constant fulfilled quantity every time an order from the retailer is received and which is equal to the retailer expected order mean. This may not be appreciated by the retailer who faces considerable variations in their market demand and thus, become reluctant to abide with the supplier's strategy.

It is also obvious from mathematical perspective that when $1 < a < 2$, the variance of the fulfilled order quantities will also be reduced to less than the variance of the original orders. However, this is not advised for consideration mainly because that when the retailer's orders happen to be large, especially when $X > \frac{a}{(a-1)} \mu > \mu$, then the proposed strategy will lead to a fulfilled quantity that is negative. Therefore, it is recommended to restrict $0 < a < 1$.

4.2 Impact on the retailer's service level

Regarding the service level at the retailer side using the new strategy of the supplier's-controlled fulfillment quantity, i.e. X_f, the probability of stock out is $P(X_f > S)$, S being the retailer's base stock level, and it is calculated below.

Here, and following [11], it is assumed that the retailer demand during the protection period is normally distributed with mean μ and standard deviation σ, so that the base stock level for the retailer is $S = \mu + k\sigma$, where k is the safety stock factor.

The probability of stock out using the above strategy of controlled fulfilled orders is

$$
\begin{aligned}
&P\{X_f > S\} \\
&= P\{X - a\,(X-\mu)\,(1_{X\leq\mu} + 1_{X>\mu}) > \mu + k\sigma\} \\
&= P\left\{\frac{(X-\mu)}{\sigma}(1-a) > k | X \leq \mu\right\} \cdot P\{X \leq \mu\} \\
&\quad + P\left\{\frac{(X-\mu)}{\sigma}(1-a) > k | X > \mu\right\} \cdot P\{X > \mu\} \\
&= P\left\{X > \mu + \frac{k}{(1-a)}\sigma | X \leq \mu\right\} \cdot P\{X \leq \mu\} \\
&\quad + P\left\{X > \mu + \frac{k}{(1-a)}\sigma | X > \mu\right\} \cdot P\{X > \mu\} \qquad (7)
\end{aligned}
$$

Note that the first part of the right-hand side of equation (7) is zero for the contradiction in the conditional probability with its conditional event. The second conditional probability event is a subset of its condition. Hence, the probability of stock out is

$$
\begin{aligned}
P\{X_f > S\} &= P\left\{X > \mu + \frac{k}{(1-a)}\sigma\right\} \\
&\leq P\{X > \mu + k\sigma\} = P\{X > S\} \qquad (8)
\end{aligned}
$$

Inequality (8) results from having $0 < a < 1$, and thus $X > \mu + \frac{k}{(1-a)}\sigma > \mu$. This reveals that the when the controlled strategy is used, the service level for the retailer will be enhanced. The larger the value of a using the new strategy, the better the service level for the retailer. This improvement in the service level for the retailer was also validated by simulation as it was revealed in section 3.2.

4.3 The impact of the control strategy on the retailer's average inventory level

To investigate the impact of the strategy under consideration on the retailer's inventory, several simulation-runs like the above were conducted over the whole spectrum of the control parameter a, i.e. from 0 to 1, with constant increments equal to 0.05. All other system parameters were maintained constant at the values assumed in section 3.2.

The average inventory level for both the original retailer order quantities and the supplier-controlled fulfillment were recorded for every simulation run and the relative increase in the average inventory level under the control strategy was calculated with respect to the average inventory level of the retailer based on uncontrolled fulfillments.

Figure 2 shows a chart that displays the profile of the percentage of the relative increases in the average inventory level when the control strategy was applied, with respect to the average inventory without control and that is in terms of the incrementing control parameters a.

It appears from this chart that the increase in the average inventory level for the retailers remains within 10% as long as the parameter a was below 0.8 and that percentage increase was less than 5% for a being less than 0.7. It is worth to remind the reader that these results are not standard and depend on other parameters assumed in the simulation. The appropriate value of this parameter should therefore be investigated case by case and simulation should be conducted to settle on the right control parameter to use.

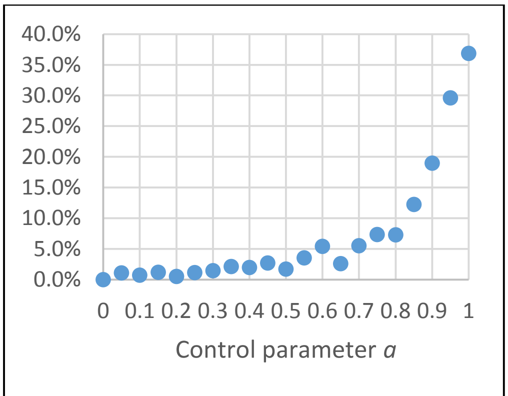

Figure 2 Percent change in the controlled inventory average

5. Conclusion

Several measures have been considered to mitigate the bullwhip effect in supply chains. Most of these strategies can be categorized under four classes: information sharing, vendor managed inventories, trust and collaboration, and supplier-controlled policies.

This paper proposed a new strategy that can be classified as a supplier-controlled strategy and provide the supplier the capability to control the order fulfillment of retailers on a one-by-one basis.

It was shown that with this strategy the variance of the fulfilled orders to retailers can be reduced while maintaining the expected mean of the original retailer orders.

At the same time, this strategy improved the service level for the retailer.

The results were validated by simulation and the impact of the control parameter on the average inventory level for the retailer was discussed. Further investigation of this strategy should address its implementation issues, especially the retailer's acceptance of having their orders fully controlled by the supplier.

In addition, future work is required for such strategy to incorporate correlated orders from a pool of retailers instead of just one.

References

[1] Ross, S. M. 2006. *Simulation*. Academic Press.

[2] Wang, X., & Disney, S.M., (2016). The bullwhip effect: Progress, trends and direction. *European Journal of Operational Research, 250* (2016), 691-701.

[3] Dejonckheere, J., Disney, S.M., Lambrecht, M.R., and Towill, D.R., (2002), Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research*. Vol. 147, 567–590.

[4] Chen, L., Lee, H. L. (2016). Modeling and measuring the bullwhip effect. Ha, A.Y., Tang, C.S., eds. *Handbook of Information Exchange in Supply Chain Management.* Springer Series in Supply Chain Management, Vol. 5 (Springer, New York), 3–26.

[5] El-Tannir, A. A., (2014). Bullwhip effect variance ratio approximations for aggregated retail orders in supply chains. *International Journal of Supply Chain Management, 3*(3), 43-48.

[6] Lee, H. L., Padmanabhan, V., & Whang, S. (1997). Information distortion in a supply chain: the bullwhip effect. *Management Science*, 43(4), 546–558.
[7] Agrawal, S. Sengupta, R.N., and Shanker, K., (2009). Impact of information sharing and lead time on bullwhip effect and on-hand inventory. *European Journal of Operational Research*, 114(1), 165-178.
[8] Choudhary, D. and Shankar, R., (2015). The value of VMI beyond information sharing under time-varying stochastic demand. *International Journal of Production Research*, 53(5), 1472-1486.
[9] Almeida, M.; Marins, F.; Salgado, A.; Santos, F.; and Silva, S., (2015). Mitigation of the bullwhip effect considering trust and **c**ollaboration in supply chain management: a literature review. *International Journal of Advanced Manufacturing Technology*, Vol. 77, 495-513
[10] Lin, Chinho; Lin, Yu-Te., (2006). Mitigating the bullwhip effect by reducing demand variance in the supply chain. *The International Journal of Advanced Manufacturing Technology*, 28(3), 328-336.
[11] Silver, E.A., Pyke, D.F., Peterson, R., 1998. Inventory Management and Production Planning and Scheduling. Wiley, New York.

Appendix: Computing the Variance of X_f

The fulfilled order amount X_f can be expressed as

$$X_f = X - a\,(X-\mu)\cdot\left(1_{X\le\mu} + 1_{X>\mu}\right).$$

Taking the variance for X_f will provide the following:

$$\begin{aligned}
&Var\left(X_f\right)\\
&= Var[X] + a^2\,Var\left[(X-\mu)\cdot\left(1_{X\le\mu}+1_{X>\mu}\right)\right]\\
&\qquad - 2a\,Cov\left[X,(X-\mu)\cdot\left(1_{X\le\mu}+1_{X>\mu}\right)\right]\\
&= Var[X] + a^2\,Var\left[(X-\mu)\cdot 1_{X\le\mu}\right] + a^2\,Var\left[(X-\mu)\cdot 1_{X>\mu}\right]\\
&\quad +2a^2\,Cov\left[(X-\mu)\cdot 1_{X\le\mu},(X-\mu)\cdot 1_{X>\mu}\right]\\
&\quad -2a\,Cov\left[X,(X-\mu)\cdot 1_{X\le\mu}\right]\\
&\quad -2a\,Cov\left[X,(X-\mu)\cdot 1_{X>\mu}\right] \qquad \text{(A-1)}
\end{aligned}$$

Now,

$$\begin{aligned}
&Var\left[(X-\mu)\cdot 1_{X\le\mu}\right]\\
&= E\left[(X-\mu)^2\cdot 1_{X\le\mu}\right] - E\left[(X-\mu)\cdot 1_{X\le\mu}\right]^2 \qquad \text{(A-2)}
\end{aligned}$$

where

$$\begin{aligned}
&E\left[(X-\mu)\cdot 1_{X\le\mu}\right]\\
&= E\left[X\cdot 1_{X\le\mu}\right] - \mu\,E\left[1_{X\le\mu}\right]\\
&= \mu'_L - \mu\,P(X\le\mu). \qquad \text{(A-3)}
\end{aligned}$$

and

$$\begin{aligned}
&E\left[(X-\mu)^2\cdot 1_{X\le\mu}\right]\\
&= E\left[X^2\cdot 1_{X\le\mu}\right] + \mu^2\,E\left[1_{X\le\mu}\right] - 2\mu\,E\left[X\cdot 1_{X\le\mu}\right]\\
&= \mu''_L + \mu^2\,P(X\le\mu) - 2\mu\mu'_L \qquad \text{(A-4)}
\end{aligned}$$

Here μ'_L, and μ'_R, are as the partial means as defined in (3)-(4), while μ''_L, and μ''_R are defined as in the following:

$$\mu''_L = E\left[X^2\cdot 1_{X\le\mu}\right] = \int_0^\mu x^2\,f(x)\,dx, \qquad \text{(A-4)}$$

$$\mu''_R = E\left[X^2\cdot 1_{X>\mu}\right] = \int_\mu^{+\infty} x\,f(x)\,dx. \qquad \text{(A-5)}$$

Note that $E\left[1_{X\le\mu}\right] = \int_0^\mu f(x)dx = P(X\le\mu)$ and therefore,

$$\begin{aligned}
&Var\left[(X-\mu)\cdot 1_{X\le\mu}\right]\\
&= \mu''_L + \mu^2\,P(X\le\mu) - 2\mu\mu'_L - \left(\mu'_L - \mu\,P(X\le\mu)\right)^2\\
&= \mu''_L - {\mu'_L}^2 + \left[\mu^2\,P(X\le\mu) - 2\mu\,\mu'_L\right]\left[1-P(X` \le\mu)\right]. \qquad \text{(A-6)}
\end{aligned}$$

With similar derivation one can also obtain

$$\begin{aligned}
&Var[(X-\mu)\cdot 1(X>\mu)] = \mu''_R - {\mu'_R}^2 +\\
&\left[\mu^2\,P(X>\mu) - 2\mu\,\mu'_R\right][1-P(X>\mu)]. \qquad \text{(A-7)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&Cov\left[(X-\mu)\cdot 1_{X\le\mu}, (X-\mu)\cdot 1_{X>\mu}\right]\\
&= E\left[(X-\mu)^2 1_{X\le\mu}\cdot 1_{X>\mu}\right] - E\left[(X-\mu)\cdot 1_{X\le\mu}\right]\cdot E\left[(X-\mu)\cdot 1_{X>\mu}\right]\\
&= 0 - \left[\mu'_L - \mu\,P(X\le\mu)\right]\left[\mu'_R - \mu\,P(X>\mu)\right]\\
&= -\mu'_L\mu'_R + \mu\left[\mu'_L P(X>\mu) + \mu'_R P(X\le\mu)\right] - \mu^2\,P(X\le\mu)P(X>\mu) \qquad \text{(A-8)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&Cov\left[X,(X-\mu)\cdot 1_{X\le\mu}\right]\\
&= E\left[X(X-\mu)\cdot 1_{X\le\mu}\right] - E[X]\cdot E\left[(X-\mu)\cdot 1_{X\le\mu}\right]\\
&= E\left[X^2\cdot 1_{X\le\mu}\right] - \mu\,E\left[X\cdot 1_{X\le\mu}\right] - \mu\mu'_L + \mu^2\,P(X\le\mu)\\
&= \mu''_L - 2\mu\mu'_L + \mu^2\,P(X\le\mu), \qquad \text{(A-9)}
\end{aligned}$$

And in the same way

$$Cov[X, (X-\mu)\cdot 1_{X>\mu}] = \mu"_R - 2\mu\mu'_R + \mu^2\, P(X>\mu). \qquad \text{(A-10)}$$

Putting all terms back in equation (A-1) and cancelling similar terms, we obtain:

$$Var(X_f) = Var(X) + (a^2-2a)\Big[\Big(\mu"_L - 2\,\mu\,\mu'_L + \mu^2\, P(X\le\mu)\Big) + \Big(\mu"_R - 2\,\mu\,\mu'_R + \mu^2\, P(X>\mu)\Big)\Big] - a\,[\mu\, P(X\le\mu) - \mu'_L + \mu\, P(X>\mu) - \mu'_R]^2 \qquad \text{(A-11)}$$

Now,

$$\mu"_L - 2\,\mu\,\mu'_L + \mu^2\, P(X\le\mu) = \int_0^\mu x^2 f(x)dx - 2\mu\int_0^\mu xf(x)dx + \mu^2\int_0^\mu f(x)dx = \int_0^\mu (x-\mu)^2 f(x)dx, \qquad \text{(A-12)}$$

and

$$\mu"_R - 2\,\mu\,\mu'_R + \mu^2\, P(X>\mu) = \int_\mu^{+\infty} x^2 f(x)dx - 2\mu\int_\mu^{+\infty} xf(x)dx + \mu^2\int_\mu^{+\infty} f(x)dx = \int_\mu^{+\infty}(x-\mu)^2 f(x)dx. \qquad \text{(A-13)}$$

while

$$a\big(\mu\, P(X\le\mu) - \mu'_L\big) + a\big(\mu\, P(X>\mu) - \mu'_R\big) = a\left[\int_0^\mu f(x)dx - \int_0^\mu xf(x)dx\right] + a\left[\int_\mu^{+\infty} f(x)dx - \int_\mu^{+\infty} xf(x)dx\right]$$

$$= -a\int_0^\mu (x-\mu)f(x)dx - a\int_\mu^{+\infty}(x-\mu)f(x)dx. \qquad \text{(A-14)}$$

Replacing the equivalents in (A-12), (A-13) and (A-14) into (A-11), we obtain

$$Var(X_f) = Var(X) + (a^2-2a)\left(\int_0^\mu (x-\mu)^2 f(x)dx + \int_\mu^{+\infty}(x-\mu)^2 f(x)dx\right) - a\left[\int_0^\mu (x-\mu)f(x)dx + \int_\mu^{+\infty}(x-\mu)f(x)dx\right]^2$$

$$= Var(X) + (a^2-2a)[\,\mu"_L - 2\,\mu\,\mu'_L + \mu^2\, P(X\le\mu) + \mu"_R - 2\,\mu\,\mu'_R + \mu^2\, P(X>\mu)] - a^2[\mu\, P(X\le\mu) - \mu'_L + \mu\, P(X>\mu) - \mu'_R]^2 \qquad \text{(A-15)}$$

In the above derivation, it was clear that by substituting $\mu'_L + \mu'_R = \mu$ the last term become zero, therefore,

$$Var(X_f) = Var(X) + (a^2-2a)[\,\mu"_L + \mu"_R - 2\,\mu\,(\,\mu'_L + \mu'_R) + \mu^2\,] - a^2[\mu - \mu'_L - \mu'_R] \qquad \text{(A-16)}$$

then by using the definitions of $\mu"_L$and $\mu"_R$, we obtain $\mu"_L + \mu"_R = \int_0^{+\infty} x^2 f(x)dx$.

Therefore,

$$Var(X_f) = Var(X) + (a^2-2a)\left[\int_0^{+\infty} x^2 f(x)dx - \mu^2\right]$$

$$= Var(X) + (a^2-2a)Var(X)$$

So finally,

$$Var(X_f) = (1-a)^2 Var(X). \qquad \text{(A-17)}$$