Responsiveness and Recovery Performance Analysis, and Replenishment System Design in Supply Chains under Large Replenishment Lead-times and Uncertain Demand

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***Abstract* -- This paper takes up the responsiveness and recovery performance of a distribution warehouse of a supply chain operating under conditions of large lead times (lags) for order replenishment, and uncertain demand (sudden demand increases). The performance metrics for measuring Responsiveness and Recovery are defined, and the dynamic performance analysis of the warehouse inventory management system under different dynamic replenishment control schemes is dealt with in detail. And from a comparative performance analysis, a good control scheme is suggested to enhance responsiveness and recovery. To further study the effect of replenishment lag specifically, a comparison of the response characteristics of some closely related supply chain conditions is then presented, which highlights the increased difficulty in controlling systems in the presence of replenishment lags compared to the others. The contributions of this study are threefold: 1) The Responsiveness and Recovery Performance Analysis of the system, 2) Replenishment System Design for enhanced Responsiveness and Recovery, and 3) Highlighting of the specific effect of replenishment lag on system performance.**

Keywords -- *Supply Chain Responsiveness, Supply Chain Recovery, Dynamic Modelling, Replenishment/Ordering Controls, Large Replenishment lags*

1. Introduction

A frequently encountered strategy in supply chain (SC) management is that of a *Responsive* chain, a fundamental requirement of which is that it should be able to respond quickly to unanticipated changes in demand ([7]). The Responsiveness strategy is suitable in scenarios with less predictable and more uncertain demands, the prime focus being on the *time taken to respond* to external disturbances. The normally accepted strategy for achieving good responsiveness is through the maintenance of adequate pipeline inventories throughout the chain, to ensure an adequate fill rate ([7]). However, when the demand suddenly increases, the presence of large replenishment lags in the system can *magnify* the effect of the demand disturbance, and result in stock-outs, with back-orders increasing dramatically. The large back-order positions can consequently ruin the responsiveness characteristics of the chain. And hence the presence of replenishment lags in SCs subject to uncertain demands can be viewed as one of the most severe forms of disruptions with respect to responsiveness performance.

Concomitant to this, an important feature of a SC is the concept of its *recovery* from disruptions and is an area of growing interest and research ([8], [28], [12], [41]). And one of the most important underlying causal factors affecting this phenomenon is the *dynamics* of the system, and consequently the recovery characteristics of the system from disruptions can be studied in detail through dynamic modelling of the system.

And hence there is a felt need to be able to study the *dynamic* responsiveness and recovery behavior of a SC operating under large replenishment lags when subjected to sudden demand disturbances. And consequently, this would help in selecting good replenishment controls to enhance the responsiveness and recovery characteristics of the chain.

Thus, the research questions that this paper seeks to answer are the following:

1) Can we predict the responsiveness and recovery behavior of the system under the conditions stated above?
2) Can we then suggest a good dynamic replenishment control to enhance responsiveness and recovery?
3) Concomitant to the above, can we also study the specific effect of replenishment lag on system behavior?

Hence the triple objectives of our paper are:

1) Firstly, to model and predict the responsiveness and recovery behavior of the system under conditions as mentioned above.
2) Secondly, to synthesize a good dynamic replenishment control to enhance responsiveness and recovery performance.
3) Finally, to study the specific effect of lag on the system, which is taken up through a

comparative analysis.

And as a spin-off his could also be of benefit for the study of resilience of a SC in future, which in essence, and implicitly involves, the recovery of the system from disruptions.

To this end, section 2 briefly reviews the relevant literature, while section 3 sets down the Methodology and the dynamic modelling framework for the study, while also defining Responsiveness and Recovery measures, and other relevant performance metrics. Sections 4 through 7 examine the dynamic behavior under various replenishment control policies, while section 8 presents a detailed comparative study and discussion of their performance. Section 9 then examines specifically the effect of lag on system performance by comparing the performance characteristics of this system with two other parallel systems closely related to this one, while section 10 translates the results down to actionable points and hence mentions the managerial implications of the study. And finally, Section 11 concludes by summarizing the main results, and mentioning the limitations of the study and scope for further work.

2. Literature Review

The literature on SC dynamics commenced with the application of servo-control mechanisms in production-control [38] and the use of system dynamic methodologies [13] subsequently. [4] and [33] provide comprehensive surveys on the subsequent use of these methods in production-inventory systems, while [36], [37] examine some of the modelling aspects. The recent books [27], [10], and [20] also cover some of the modeling concepts. Concomitantly, there has been a large amount of research work recently pertaining to optimal inventory policies and the bullwhip effect and its mitigation on the one hand, as well as on SC coordination and coordination mechanisms using pricing and profit-sharing contracts on the other, both of which we do not cite herein, since our focus is primarily on the responsiveness and recovery characteristics (e.g. [3], [11], [34], [42], [44]). On the topic of SC responsiveness and recovery, some of the most recent papers, which we have listed in the references, cover a wide range of topics, and use a wide range of methods ranging from conceptual formulations, statistical analyses, case analyses, simulation, and DEA. But there appears to have been less work on the dynamic mathematical modelling aspects of SC disruptions and their analysis, thereby pointing to a discernible gap in the literature on responsiveness and recovery. And since the predictive capability of mathematical models make them an important constituent in any topic, this paper attempts to fill this glaring gap using dynamic mathematical modelling methods.

One of the recent papers [39] takes up control-theoretic modelling using block-diagrams and transfer function methods (in the Laplace-transformed domain) to define and quantify resilience in a SC. And after a thorough analysis, it concludes that most control strategies are *not robust to lead-time disturbances*. Its conclusions also reinforce the need for better controls for enhancing the performance of SCs under such conditions. A recent paper on inventory control in a dynamic supply chain system considering supply-price trade-off also uses closed-loop control theory ([2]), while another ([1]) has used optimal control theory as a means to mitigate the bull-whip effect in a feedback control framework through extensive numerical computations. Though the latter two would pertain more to efficient chains, however, they reinforce the applicability of such dynamic modelling methods in SC analysis. Our paper uses similar methods but deals with responsive chains, focusing on responsiveness and recovery.

In one of the recent papers ([21]) the interactions between sustainable and resilient SCs have been explored, and one of its important conclusions, amongst others, is that *response effectiveness with regard to disruptions is much sought after by diverse SCs globally.* These findings further reinforce the need to have better controls to enhance responsiveness and recovery, which is one of the important objectives of our study.

Some recent work on the disruptive effects of lead-time on SC performance are [17] on a simulation-based study showing the major impact of lead time variability on SC inventories, stock-outs and order variances all through the chain on the one hand, and [5], [6], [23], and [30] on the other, which also deal with variable lead-times; however their quest is for *optimal* inventory control policies in such contexts as is pertinent to *efficient* chains. The papers [24] and [23] deal with optimal expediting policies and optimal contingency policies respectively in the face of disruptions, while [26] looks at lead-time management using expediting hubs. While [19] deals with management of supply disruptions using incentives for capacity restoration, [40] and [18] take up disruptions with unreliable endogenous supply processes, and reliable/unreliable suppliers respectively.

The papers above analyze the effects of lead-time using statistical methods on the one hand, and a wide range of mitigation mechanisms like incentives for capacity restoration, expediting hubs, more reliable suppliers etc. on the other. Whereas a dynamic mathematical model of the system under large lead-times and its dynamic *analysis*, was not found to have been taken up in detail in the literature. And this paper attempts to bridge this apparent gap.

It is only recently that attention and research has now started focusing on the *dynamic control* of responsive SCs to improve their response times ([31], [27]). A comprehensive review paper on this topic is [14] which focusses on control methods and identifies dynamic modelling as an important facet of the study of SC performance. A few recent papers ([25], [32], [39]) have treated some aspects of dynamic system behavior under conditions of zero lag, whereas our paper takes up the important case of SCs under *large non-zero lags*.

3 Methodology: The Dynamic Modelling Framework and Design Stipulations

3.1 The Dynamic Modelling Notation

In developing the dynamic model equations we make use of *deviation* variables as is usually done in modelling of dynamical systems (e.g. [31], [39]):

$x_i(k)$ is the inventory *deviation* (from its design/nominal value) at time k, at stage i of the chain

$q_i(k)$ is the material flow *deviation* (from its design/nominal value) in period (k-1, k], into stage i

$r_3(k)$ is the *deviation* (from the predicted) in demand observed at the warehouse in (k-1, k].

With: i = 1, representing (the raw material) the upstream end of the production facility

i = 2 representing (the finished goods) the downstream end of the production facility

i = 3 representing (the finished goods) the warehouse.

The dynamic equations of the system can then be written in terms of these deviation variables as:

$$x_i(k+1) = x_i(k) + q_i(k+1) - q_{i+1}(k+1) \quad (3.1)$$

Where for i = 3, $q_4(k) = r_3(k)$, is the demand outflow from the warehouse.

The standard initial conditions are: $\{(x_i(k), q_i(k), r_3(k)) = (0,0,0), k \leq 0\}$. The control variables are the replenishment flows, through which control is exercised over the system.

In our study, we take the demand disturbance to be a sudden and sustained increase in demand off-take, while the *replenishment lag* is presumed to be *pre-existing* in the system. The demand disturbance in our system is represented by a Heaviside step function of magnitude b_0 , and a random disturbance term superimposed on it, given by:

$$r_3(k+1) = b_0 H(k) + \varepsilon(k+1), \quad \text{where} H(k) = \begin{bmatrix} 1, k \geq 0 \\ 0, k < 0 \end{pmatrix} \quad \text{and,} \quad \varepsilon(k) \approx WN(0, \sigma^2) \quad (3.2)$$

, with $\varepsilon(k)$, the stochastic component of the disturbance, usually taken to be a White Noise process.

The replenishment *lag* is measured by the number of periods of delay between the period of the physical arrival of the consignment, and that immediately succeeding the period of order initiation. This convention is based on that generally followed in practice as well as in the literature. And hence the lag is taken as zero if the ordered consignment arrives within the next period (instantaneous delivery being relatively rare in practice).

3.2 Performance Metrics for Responsiveness, Recovery, and Replenishment System Design

In our paper, we take the *Lag-specific back-orders* in unit-periods per unit of lag as defined below, as an inverse measure of responsiveness; i.e. the lower its value, the higher would be taken to be the responsiveness. This is defined as under:

Lag-s*pecific Back-orders*

$$= \frac{-\sum_{k=0}^{t} min(0, x_i(k))}{l_i + 1} \quad (3.3)$$

in stage i of the chain, measured in number of *unit-periods per period of replenishment lag*, and where l_i is the lag present in stage i of the chain. The numerator is the cumulative back-orders till restoration to normalcy, and is divided by the number of periods of lag present in the system. This measure has an inverse relationship with the fill-rate. This measure can be interpreted as the Back-orders *per period of lag*.

Note1: We divide by $(l_3 + 1)$ rather than by l_3 to be consistent with our definition and measurement of lag, since it is presumed that the earliest delivery of consignments ordered would be in the succeeding period (instantaneous delivery being relatively rare in practice).

We next define the *Lag-adjusted Recovery Time* which we take as an inverse measure of Recovery.

Lag-adjusted Recovery Time: The Recovery Time *less* the number of periods of lag in the system, i.e.

Lag-adjusted Recovery Time = Recovery Time – (Number of periods of lag in system) (3.4)

where the Recovery Time is the time taken as the time to restore the system to within 90% of its normal and stable operating levels. And since the *state variable* of the system is its inventory level, we define the Recovery time as under:

Recovery Time: The time taken to restore the inventory level of the system to within $\pm 10\%$ of its normal (nominal) operating levels, *and maintain it thereafter*, with time being measured from the occurrence of disruption/disturbance.

We can note that to recover from a disruption, the system would need a duration of time at least equal to the number of periods of lag in the system, and hence reducing the Recovery time by the lag would yield the *intrinsic* recovery time of the system which is the quantity of critical interest to us in recovery performance analysis.

In our paper we evaluate Responsiveness and Recovery performance using the two metrics above.

We next look at the other response characteristics which would impact replenishment system design, since our objective is also to synthesize a good

replenishment control scheme for enhanced performance.
The response of the system to the demand disturbance will have two parts:
1) The deterministic part: the *mean response,* which would *characterize* the system behavior, and,
2) The stochastic part, representing the random fluctuations that could occur even after the system is restored to normalcy, which is characterized by the inventory variance, and results for which are available in the literature.
Due to, and subsequent to the perturbation of the system by the sudden jump in demand, the inventory levels decrease initially and fluctuate before they are restored to their normal levels again. In this context, the *dynamic performance metrics* for the *mean response* that are of interest from the point of view of replenishment system design would be the following ([43], [31]):
a. The "trough value" or the lowest dip in the inventory levels, which determines the base-stocks to be carried in the system to prevent disruptions (the 'undershoot').
b. Permanent depletion of the inventory level (the "offset") if any, which indicates whether the system is restored to its original level, or not.
c. The amplitude of fluctuations of the inventory, which we would like to damp out as rapidly as possible to restore the system to steady operation as quickly as possible.
d. The extent of time the inventory level stays depleted (in the negative region), which can be taken as an indirect measure of the stock-out risk; and the larger the magnitude and extent of time in negative inventory region, the larger would be taken to be the implied risk.
Additionally, for the above characteristics of the mean response to be meaningful, we would also *require the inventory variance*, which is commonly taken as a measure of the robustness of the system to random variations, to be *bounded*, results for which are available in the literature. Consequently, we henceforth focus our attention on the mean response.

3.3 Dynamic Replenishment Controls, Design Objectives, and Design Stipulations

In most supply systems, the replenishment control action is triggered by the inventory and demand levels at the warehouse ([35], [43], [31]). Thus, in most cases the replenishment flow is given by a function of the latest available/observed set of inventory and demand deviations, as under:
$q_i(k+1) = f(x_3(k-1-l_i), r_3(k-1-l_i), x_3(k-2-l_i), r_3(k-2-l_i), x_3(k-3-l_i), \ldots),$
for i = 1,2,3 (3.5),
where, the l_is are the lags in the different stages of the system. When the value of the *lag is positive*, any *increase in demand* will pull down the inventory levels leading to inventory fluctuations both at the warehouse as well as upstream units. It thus becomes essential in a responsive chain to *specifically design* the replenishment system to be able to *handle demand increases in the presence of delays and replenishment lags*.
The number of periods of lag as well as the magnitudes of demand increases are both environmental parameters which the designer has no control over. And hence, *the objective of the design* would be to *handle as high a lag, and simultaneously as high a magnitude of demand increase,* as would reasonably be possible to design for. In our paper, we analyze the system behavior for up to as high a lag as is found tractable by elementary methods for a step increase in demand. O*ur objective is to choose a control scheme that would be capable of handling arbitrarily high lags in situations of sudden increases in demand.*
We focus our attention hereafter on the downstream end of the chain, viz. the warehouse, which is the interface between the SC and its market, and whose performance determines the operational performance of the entire SC. Upstream units can be analyzed in likewise manner, using the inflow to the succeeding downstream units as the "demand" for the immediately preceding upstream unit in the chain.
We now derive the Initial Conditions (ICs) for the warehouse system.
Since the lag is pre-existing in the system, the increased demand would keep pulling down the inventory at the warehouse *by the quantum of increase in demand,(* 'b_0' units every period), till the period (l_3+1), only after which the system begins to feel the effect of the additional replenishment flow (due to the replenishment control action). Thus the ICs are:
$\{x_3(0) = 0, x_3(k) = -kb_0$ in $0 \le k \le l_3, x_3(l_3+1) = -l_3 b_0 = x_3(l_3+2)\}$ } (3.6).
The last equality in (3.6) is by virtue of the starting condition: $x_3(0) = 0$.
Under the conditions stated above, we first derive the responsiveness and recovery behavior of the warehouse inventory system under the three conventional forms of dynamic controls, the Proportional controls (P(I)), Proportional-integral controls (PI(I)), and the Proportional-integral-derivative controls (PID(I)) respectively, where the "(I)" within the parentheses denotes "inventory-triggered". Subsequently we take up the Moving Average (MA(ID)) controls, where likewise, "(ID)" within parentheses denotes "Inventory and Demand-triggered". We then conduct a detailed performance analysis of these controls and then suggest a good scheme with enhanced responsiveness and recovery performance.

In this, our *Design Stipulations* for the selected control will be two-fold, as under:

1) Firstly, it should *reduce both Lag-specific Back-Orders and the Lag-adjusted Recovery Time* (thereby enhancing both responsiveness and recovery performance)

2) Secondly, the control design should be *mathematically tractable* for any *arbitrarily large lag*.

We now proceed with the performance analysis of the four control schemes keeping these two stipulations in mind.

4. Proportional Replenishment Control (P(I)-Systems)

In this simple scheme, the replenishment order quantities at each stage and in each period are set *proportional* to the latest observed inventory deviations at the warehouse, i.e.

$q_i(k+1) = K_i x_3(k-1-l_i)$ for i = 1,2,3 (4.1)

Where K_i are the 'proportionality' factors (and hence the term 'proportional' control).

Substituting for the replenishment flows, the dynamic equation of the warehouse is given by:

$x_3(k+1) - x_3(k) - K_3 x_3(k-1-l_3) = -r_3(k+1) = -b_0 H(k) + \varepsilon(k+1)$, valid in $k \geq l_3 + 1$ (4.2)

Using the forward shift operator E, defined by: $Ex(k) = x(k+1)$, the equation can be written in standard Operator form as:

$(E^{l_3+2} - E^{l_3+1} - K_3)x_3(k) \equiv -E^{l_3+1} b_0 H(k) + \varepsilon(k+_3 l+2) \equiv -b_0 + \varepsilon(k+l_3+2)$ (4.3) , valid in $k \geq 0$.

This is a linear difference equation (LDE) of order (l_3+2), with the (l_3+2) initial conditions derived above. Hence, we have the *Deterministic LDE* for the mean response, and the *Stochastic Difference Equation (SDE)* for the stochastic part of the response, given respectively by:

$(E^{l_3+2} - E^{l_3+1} - K_3)x_3(k) \equiv -b_0 H(k+1) = -b_0$, valid in $k \geq 0$ (4.4a) (for the mean response)

$(E^{l_3+2} - E^{l_3+1} - K_3)x_3(k)^{stoc} \equiv -\varepsilon(k+l_3+2)$, valid in $k \geq 0$ (4.4b) (for the stochastic part).

We focus our attention on the deterministic LDE, the solution of which yields the mean response, which governs the response characteristics.

In the literature the SDE is used in computing the limiting inventory variance to essentially check for its bounded-ness. In our paper we take the results on inventory variance from the literature. In a specific case for which it is not available, we derive them in the Appendix.

4.1 Single period Lag

For single period lag, i.e. $l_3 = 1$, eqn. (4.4a) reduces to: $[E^3 - E^2 - K_3]x_3(k) \equiv -b_0$ valid in $k \geq 0$, with ICs: $\{x_3(0) = 0, x_3(1) = -b_0, x_3(2) = -2b_0\}$. Elementary analysis yields the region of stability for the system as: $(-0.619 < K_3 < 0)$, and instability in: $K_3 < -0.619$, and $K_3 \geq 0$.

The best damping is obtained for $K_3 = -(4/27)$. We also look at the maximum permissible value of $K_3 = -0.619$ as an extreme case.

The system response for maximum damping for $K_3 = -(4/27)$ and the extreme case are given respectively by:

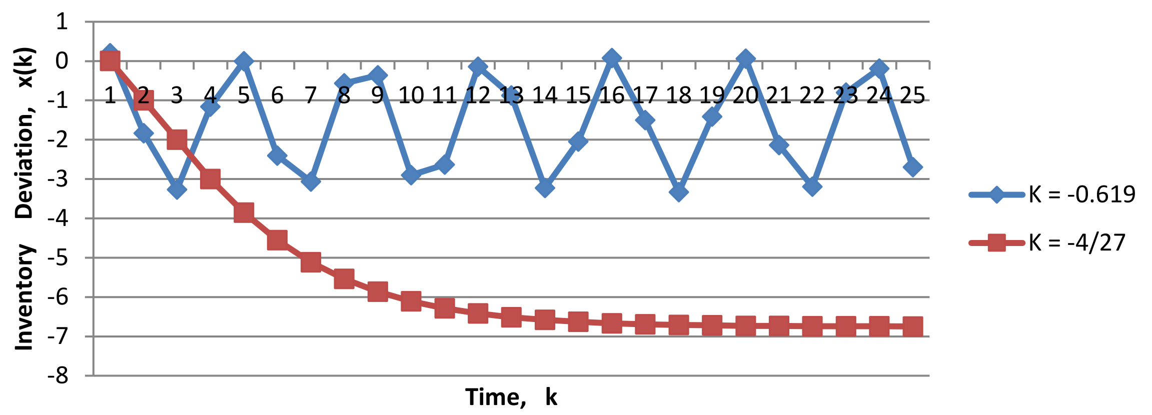
$x_3(k)/b_0 \equiv -6.75 + (0.0833)(-1/3)^k + (6.667 + 2k)(2/3)^k$, valid in $k \geq 0$, and (4.5)

$x_3(k)/b_0 \equiv -1.6155 + (0.0988)(-0.619)^k + 1.7184 Cos(0.629k + \varphi)$, valid in $k \geq 0$, with $tan\, \varphi = -0.6161$ (4.6)

The RHSs of the equations above are the responses for unit step demand. And in both equations the first term has the value $-1/(|K_3|)$, and gives the offset value in eqn. (4.5) , and the center-line of oscillations in eqn. (4.6).

Thus for smaller values of the control parameter K_3 , the system though stable has a very high offset, while for larger values of K_3 , it has high amplitude perpetual oscillations, thus showing a *trade-off between offset and amplitude of oscillations.* The response of the system is shown in Fig. 1 for the two values of $K_3 = -4/27, K_3 = -0.619$, for unit step demand. The value of total back-orders (and hence also the Lag-specific Back-orders) increases without limit as time increases in the first case, and oscillates in the second. The recovery time (and hence also the Lag-adjusted Recovery Time) becomes infinite in both cases, thereby leading to loss of responsiveness and recovery in both. The limiting inventory variances are obtained as: $lim_{k \to \infty} var(x_3(k)) = 6.05\sigma^2$ for $K_3 = -4/27$, and $4.243\sigma^2$ for $K_3 = -0.619$, which though low are of little significance due to the poor performance characteristics of the mean response.

Fig. 1 P-Control



4.2 Two-period Lag and Higher

For this case, the system equation is:

$$[E^4 - E^3 - K_3]x_3(k) \equiv -b_0 - \varepsilon(k+4) \qquad (4.7)$$

The stability condition for $l_3 = 2$ is : $-1/2 < K_3 < 0$. However, determining the exact values of the four roots of the LHS Operator analytically or computationally becomes cumbersome, since all four roots are complex. Representing the complex roots as $x + jy$ yields the system of non-linear equations in x and y (by equating the real and imaginary parts to zero separately) as:

$$x^4 - 6x^2y^2 + 4y^4 - x^3 - 3xy^2 - K_3 = -b_0 \qquad (4.8)$$

$$4x^3 - 4xy^3 - 3x^2y - y^3 = 0 \qquad (4.9)$$

which requires a two-dimensional Newton-Raphson procedure to solve for the roots *for each value of* K_3 *individually*. For a 3-period lag, the analysis becomes even more complicated and intractable by elementary methods, and we do not take it up here.

Thus, in summary, the P(I) control shows poor responsiveness and recovery performance even for lag of a single period, and also exhibits design intractability for lags beyond a single period.

5. Proportional-integral control schemes: PI(I) systems

In these schemes, the replenishment order-control flows are given by: $q_3(k+1) \equiv K_p x_3(k - l_3 - 1) + K_c \sum_{m=0}^{k-l_3-1} x_3(m)$ (5.1)

where the second term is the 'integral' component of the control (summation of the inventory deviations in our discrete-time system). Substituting for the control flows, the equation for the warehouse is given in standard form by:

$$x_3(k + l_3 + 2) - x_3(k + l_3 + 1) - K_p x_3(k) - K_c \sum_{m=0}^{k} x_3(m) \equiv -r_3(k + l_3 + 2) \equiv -b_0 \quad \forall k \geq 0 \qquad (5.2)$$

This equation is easier solved by Transform methods, and hence we make use of the Z-transform below.

5.1 Single-period Lag

For single period lag and step demand, taking the Z-transform and using the ICs, eqn. (5.2) reduces to:

$$[z^4 - 2z^3 + z^2 - (K_p + K_c)z + K_p]X_3(z) = -b_0 z \qquad (5.3)$$

The solution for highest damping is obtained as:

$$x_3(k)/b_0 = \\ = -\{-(0.6473)(-0.366)^k + (0.6473)(0.79)^k + (0.8194)k(0.79)^k + (0.6931)k(k-1)(0.79)^k\} \qquad (5.4).$$

The trough value is -4, or four times the magnitude of the demand disturbance and the response oscillates about a center-line that gradually increases to zero. The damping rate is: $O(0.79)^k$ which is very low. The total back-orders till restoration to normalcy is -30 unit-periods approximately, and the Lag-specific Back-Orders 15 which is moderately low, showing moderate responsiveness. The recovery time is 25 periods, and Lag-adjusted Recovery time is 24 periods showing slow recovery. The limiting inventory variance is: $lim_{t\to\infty} var(x_3(k)) = 3.93\sigma^2$.

5.2 Two-period Lag

The transform of equation (5.2) with $l_3 = 2$ yields:

$$[z^4 - z^3 - K_p - K_c \frac{z}{(z-1)}]X_3(z) = -\frac{b_0 z}{(z-1)} \qquad (5.5)$$

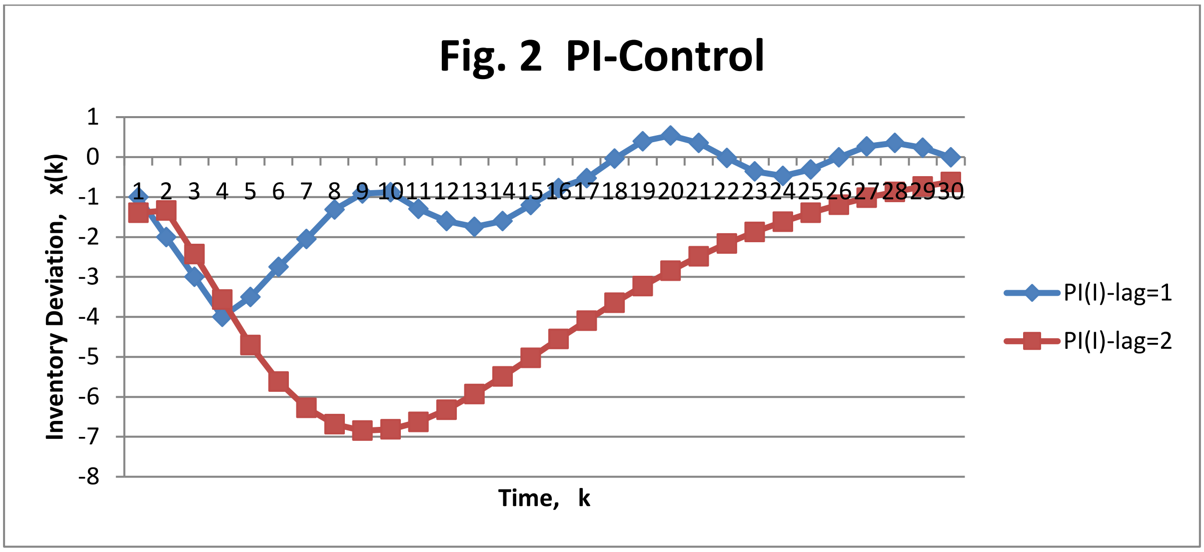
The solution for the best damping is obtained as:

$$x_3(k)/b_0 = (45.03 + 4.05k)(0.8)^k - 44.41(0.9)^k + (0.431)^k(-0.618 Cosk\theta + 0.4 Sink\theta) \qquad (5.6)$$

The response has an undershoot of -6.85, and then rises gradually to zero at the rate $(0.9)^k$, which is very slow. The total backorders till restoration of normalcy is approximately -139 unit-periods, and Lag-specific back-order measure is 70 unit-periods per period of lag, which is very large indicating low responsiveness. The Lag-adjusted recovery time increases to above 33 periods, showing very slow recovery. The limiting inventory variance in this case is $180.91\sigma^2$, which is quite large.

The responses of the system for single and two period lags are shown in Fig. 2 for unit step demand, showing a marked deterioration in the response with increase in lag.

For lags beyond two periods, the design of the system becomes difficult and intractable by elementary methods and is not taken up here.



Summarizing, we thus find that the PI(I) controls also show low responsiveness and slow recovery for lags of above a single period, and also display design intractability for lags beyond two periods.

6. Proportional-integral-derivative type of replenishment controls

In the PID scheme, the control flows are given by

$q_i(k+1) \equiv K_i x_3(k-1-l_i) + K_c \sum_{m=0}^{k-1-l_i} x_3(m) + K_d^i(x_3(k-1-l_i) - x_3(k-2-l_i))$ (6.1)

where the third term is the 'derivative' component of the control (the difference of the two latest available inventory deviations in our discrete-time system).

Hence substituting for the control flows, the warehouse equation is given by

$x_3(k+3+l_3) \equiv x_3(k+2+l_3) + K_3 x_3(k+1) + K_c \sum_{m=0}^{k+1} x_3(m) + K_d^i(x_3(k+1) - x_3(k)) - r_3(k+3+l_3)$

(6.2) .

6.1 Single Period Lag

For single period lag and system ICs, the transform of eqn. (6.2) yields: $[z^4 - z^3 - K_p z - K_c \frac{z^2}{z-1} - K_d(z-1)]X_3(z) = -z^4 R_3(z) = z^3 \frac{b_0 z}{z-1}$

(6.3)

The solution for best damping is obtained as:

$x_3(k)/b_0 = [\{0.338 - 1.945k - .248k(k-1) + 0.001k(k-1)(k-2)\}(0.645)^k - 0.338(-0.58)^k]H(k-1)$ (6.4) .

The response has an undershoot of -2 and gradually increases to zero at the rate $O(0.645)^k$, which is moderately fast. The Lag-specific backorders till restoration to normalcy is -14 unit-periods, which is moderately low, indicating moderately high responsiveness. The recovery time is about 15 periods, and the Lag-adjusted Recovery Time is 14 periods, showing moderate recovery performance.

The limiting inventory variance is obtained as: $lim_{k\to\infty} var(x_3(k)) = 4.164\sigma^2$.

6.2. Two Periods Lag.

For this case, the transform of the warehouse equation is:

$[z^6 - 2z^5 + z^4 - z^2(K_p + K_c + K_d) + z(K_p + 2K_d) - K_d]\frac{X_3(z)}{z-1} = -z^4 \frac{b_0 z}{z-1}$ (6.5)

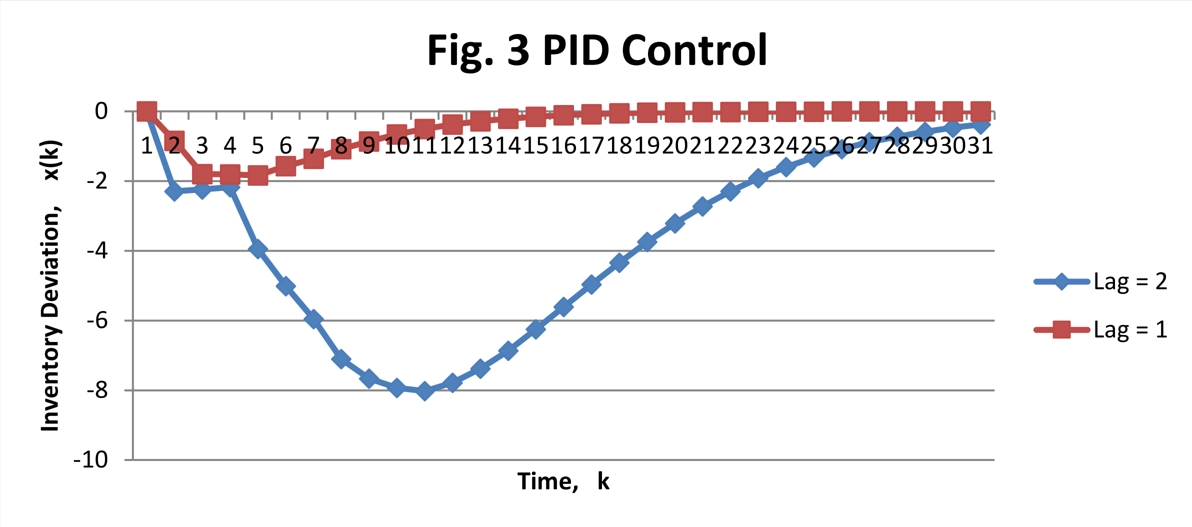
The response for best damping is then obtained as:

$x_3(k)/b_0 = [-1.58 - 0.98k + 0.42k^2 - 0.23k^3](0.725)^k + [1.58Cosk\theta + 0.003Sink\theta](0.673)^k$,

$\theta = -0.84$radians (6.6) .

The response has an undershoot of -8, and exhibits low amplitude damped oscillations about a center-line that gradually increases to zero at the rate of $(0.725)^k$, which is fairly slow. The total backorders till restoration to normalcy is -116 unit-periods, with the Lag-specific back-order measure being 39 unit-periods per period of lag which is very large, indicating loss of responsiveness. The recovery time is 33 periods, and the Lag-adjusted Recovery time is 31 periods, showing very slow recovery. The limiting inventory variance is $15.58\sigma^2$.

The response of the system for single and two period lags is shown in Fig. 3 for unit step demand.



For lags of three periods and higher, design of the system again becomes difficult and intractable by elementary methods and is not taken up here.
Thus, in summary, the PID controls show moderate responsiveness and recovery for up to a single period of lag, but poor performance thereafter. And they also exhibit *design intractability* for lags beyond two periods.
Additionally, in all the above studied controls (P(I), PI(I), and PID(I)) the inventory stays in the *negative region all through* and for long periods of time, thereby implying *high stock-out risk*.

7. Moving-Average-Type Controls

Here the control flows are set to a weighted Moving Average (MA) of the latest available inventory deviations up to a certain number (say r) periods back, r being the order of the MA, and thus being given by:

$q_i(k+1) = \sum_{l=1}^{r} K_l^3 x_3(k-1-l_3)$
(7.1)

Where the K_l^3 are the weights, which are the control parameters and can be set by us.
The advantage of these MA control schemes is that they have up to r control parameters, which can be tuned by us, and hence can be expected to perform better than the previous controls. An additional advantage is that in this type of MA control scheme, the control can be designed easily using elementary methods as shown below for *any arbitrarily high value of lag*, l_3. And the system response can be derived as a function of the lag l_3 (thus exhibiting design tractability for any high value of lag).
However, these controls are known to have an offset. And hence, to improve their performance and speed up the response, the control is usually made more *pro-active* by adding a demand trigger term to the control flows and is shown below.

7.1 The Conventional MA(ID) Control Scheme

The simplest control scheme uses two inventory trigger terms and a single demand trigger term in the MA(ID) control, setting the control flow as:

$q_3(k+1) \cong K_1^3 x_3(k-1-l_3) + K_2^3 x_3(k-1-l_3-1) + K_3^0 r_3(k-1-l_3)$ (7.2)

And hence the system equation is given by

$[E^{l_3+3} - E^{l_3+2} - K_1^3 E - K_2^3] x_3(k) \equiv -E^{l_3+2} r_3(k+1) + K_3^0 r_3(k+1)$ (7.3)

Under the condition: $K_2^3 = -K_1^3$, the LHS Operator and the LDE simplifies to:

$(E-1)(E^{l_3+2} - K_1^3) x_3(k) \equiv -E^{l_3+2} r_3(k+1) + K_3^0 r_3(k+1)$ (7.4)

The solution for our system with a unit step input is then given by:
For Case1: $l_3 + 2 =$ odd:

$$x_3(k)/b_0 \equiv C_0 + \frac{K_3^0 - 1}{1 - K_1^3} k + C_2 \rho^k - \sum_{n=1}^{\left\lfloor \frac{l_3+2}{2} \right\rfloor} \rho^k \{A_n Cos(\tfrac{2\pi nk}{l_3+2}) + B_n Sin(\tfrac{2\pi nk}{l_3+2})\} \quad (7.5a)$$

For Case2: $l_3 + 2 =$ even:

$$x_3(k)/b_0 \equiv C_0 + \frac{K_3^0 - 1}{1 - K_1^3} k + C_2 \rho^k + C_3(-\rho)^k - \sum_{n=1}^{\frac{l_3+2}{2}-1} \rho^k \{A_n Cos(\tfrac{2\pi nk}{l_3+2}) + B_n Sin(\tfrac{2\pi nk}{l_3+2})\} \quad (7.5b)$$

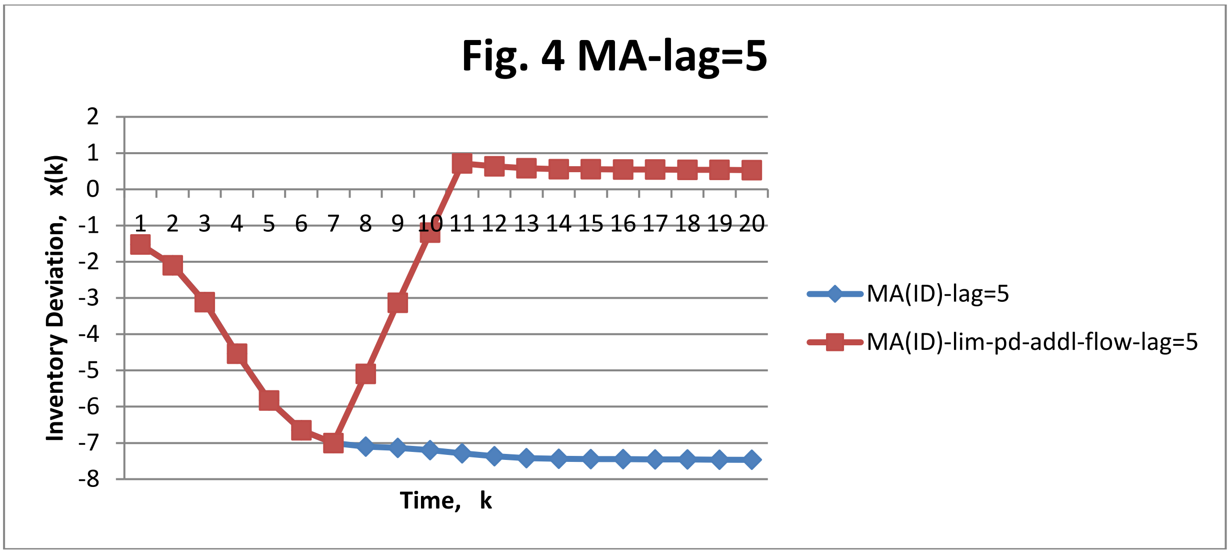
with $\rho = (K_1^3)^{1/(l_3+2)}$. We get a *stable solution only for* $K_3^0 = 1$, with offset of C_0 , and *damping rate of* $O(\rho)^k$, *which can be controlled by us* by a suitable choice of $\rho = (K_1^3)^{1/(l_3+2)}$.
The constants $\{C_0, C_2, C_3, A_n, B_n\}$can be evaluated using the initial conditions: $\{x_3(k) \equiv -kb_0$, for $k = 0,1,2,\ldots,l_3+2\}$. We can however note the presence of an offset of $-b_0 C_0$for this control also.

For a numerical illustration, we take $l_3 = 5$ and $K_3^1 = 1/16$, yielding $\rho = (1/16)^{1/7} = 0.673$.
For a unit step demand disturbance the ICs are obtained as: $(x_3(0) \quad \dots \quad \dots \quad \dots \quad x_3(7)) = (0 \quad -1 \quad -2 \quad -3 \quad -4 \quad -5 \quad -6 \quad -7)$ (7.6a),
And for a step disturbance of magnitude b_0 ,
$(x_3(0) \quad \dots \quad \dots \quad \dots \quad x_3(7)) = b_0(0 \quad -1 \quad -2 \quad -3 \quad -4 \quad -5 \quad -6 \quad -7)$ (7.6b)
From which we obtain the vector of coefficients as $(C_0, C_2, A_1, B_1, A_2, B_2, A_3, B_3) = (-7.471, 13.930, 3.591, 3.384, 2.381, -0.874, 0.487, -1.041)$,
and hence the complete response as :
$x_3(k)/b_0 = -7.47 + 13.93(0.673)^k - (0.673)^k\{3.58Cos(2\pi k/7) + 3.38Sin(2\pi k/7) + +2.38Cos(4\pi k/7) - 0.874Sin(4\pi k/7) + 0.487Cos6\pi k/7) - 1.04Sin(6\pi k/7)\}$ (7.7)
The response is shown in Fig. 4 for unit step demand. The system inventory depletes rapidly to its steady negative level of – 7.47 units ($-7.47b_0$ for a step input of magnitude b_0 units), showing complete lack of recovery. Though the damping rate is reasonably high being given by $(0.673)^k$, the total back-order position would obviously increase without limit, resulting in loss of responsiveness. The implied stock-out risk would also be high, both of which are due to the (-)ve offset.
Hence to eliminate the offset a modification is introduced as is shown below.

7.2 MA(ID) Control with Recovery Flow

A simple option used is the addition of a *recovery flow* which is essentially a step-up flow to the MA(ID) control flows for a limited period, which would be capable of restoring the system to its normal levels within a desired number of periods or within a *desired recovery time*. The control flows are thus set as:
$q_3(k+1) \cong K_1^3 x_3(k-1-l_3) + K_2^3 x_3(k-1-l_3-1) + K_3^0 r_3(k-1-l_3) + \delta(k)$ (7.8)
Hence, the demand-triggered component of the control flow is set to $K_0^3 r_3(k-1-l_3) + \delta(k)$ for a limited period of time, and thereafter, restored to the normal MA(ID) control flow.

For our system with a step input of magnitude b_0, the system response is then given by:
For Case1: $l_3 + 2 =$ odd:
$x_3(k)/b_0 \equiv C_0 + \frac{K_3^0+\delta(k)-1}{1-K_1^3}k + C_2\rho^k - \sum_{n=1}^{\left\lfloor\frac{l_3+2}{2}\right\rfloor} \rho^k\{A_n Cos(\frac{2\pi nk}{l_3+2}) + B_n Sin(\frac{2\pi nk}{l_3+2})\}$ (7.9a)
For Case2: $l_3 + 2 =$ even:
$x_3(k)/b_0 \equiv C_0 + \frac{K_3^0+\delta(k)-1}{1-K_1^3}k + C_2\rho^k + C_3(-\rho)^k - \sum_{n=1}^{\frac{l_3+2}{2}-1} \rho^k\{A_n Cos(\frac{2\pi nk}{l_3+2}) + B_n Sin(\frac{2\pi nk}{l_3+2})\}$ (7.9b) with $\rho = (K_1^3)^{1/(l_3+2)}$.
We get a stable solution for $K_3^0 = 1$, with the same damping rate of $O(\rho)^k$, with the offset term now becoming $C_0 + \frac{K_3^0+\delta(k)-1}{1-K_1^3}k$. The offset can now be controlled by setting $\delta(k)$ to the appropriate sequence. For example, if it is desired to restore the inventory level within 'n' periods, the magnitude of *recovery flow* is set as: $\delta(k) = -C_0(1-K_1^3)/n$, for $0 < k \le n$, and zero thereafter , i.e. $\delta(k) = [-C_0(1-K_1^3)/n]\{1 - H(k-n)\}$, where H(.) is the unit Heaviside step function. This will pull up the inventory deviation to zero by the n^{th} period in a smooth and uniform manner.
The response is shown in Fig. 4 for n = 4 and $l_3 = 5$. The inventory level is pulled up to (+)ve levels rapidly within 4 periods (or as designed by the planner), with total back-orders of -40 even for a high lag of 5 periods, and Lag-specific back-orders of 8 unit-periods per unit of lag, showing a good degree of responsiveness. The *recovery time is as low as 10 periods even for a high lag of 5 periods, with a Lag-adjusted Recovery time of 5 periods only, showing rapid recovery (good recovery performance).*
The implied stock-out risk is also hence low. The limiting inventory variance is $20.54\sigma^2$.
The managerial significance of this result is that the MA(ID) control scheme with a recovery flow could be built into the warehouse management system software, thereby achieving a good degree of *in-built* responsiveness and recovery capability in the warehouse system.



8 Comparative Performance and Discussion

We now present the comparative the performance of the above replenishment control policies at a glance in the tables below.

Table 1 Comparative performance of different Replenishment Policies

Sl No	Type of Control	Centre-line of Osclln *(Avg. Inv. Level)*	Ampl. of Osclln. *(damping of Fluctuation)*	Trough/ Offset value *(base stock Requirement)*	Fracn. Of time in (-)ve Regn. *(Stock-out Risk)*	Lag-specific Back-Orders *(Responsiveness)*	Recovery Time and Limitation on control parameter Settings *(Design Tractability)*	Remarks and max. lag tractable by elementary methods
1	Proportional to Inventory deviations P(I)	*Very low* (Fluctuates about a centre-line of $-1/\lvert K_3\rvert$, -6.75 to -1.65)	For K_3 : -0.619: Perpetual (ampl 1.78) fluctuations) For K_3 = -4/27: damping of $(0.67)^k$	Very low trough value: $-1/\lvert K_3\rvert$. For $l_3=1$: -6.75 to -1.65 Can have high (-)ve offset	Entirely in (-)ve region, and *high stock-out risk*	Infinite. Complete loss of control and *very low responsiveness*	Infinite recovery time; For $l_3=1$: Stability only for: $-4/27 < K_3 < -0.619$ $l_3=2$: $-1/2 < K_3 < 0$ *High Design Intractability beyond lag of 1 period*	Near perpetual high ampl. oscillations, OR High (-)ve Offset, Trade-off between offset and damping *Very Slow Recovery; High design Intractability* for lag beyond 1 period
2	Proportional -integral controls PI(I)	(-)ve initially, and slowly pulled up to zero	*Low damping* For $l_3=1$: @ $(0.79)^k$ For $l_3=2$: @ $(0.944)^k$	For $l_3=1$: Trough of - 4 For $l_3=2$: Trough of -6.85 *But has zero offset*	Entirely in (-)ve region *(high stock-out risk)*	Fairly high for $l_3=1$: -15 *High* For $l_3=2$: -70	*Recovery Time: 25 for lag =1, 33 for lag =2; High Design Intractability beyond lag of 2 periods*	Fluctuations damped out and system restored to its original state, but slowly; *Slow Recovery; High Design intractability for lags beyond 2*

3	Proportional -integral-derivative controls PID(I)	-- do --	*Low damping* For $l_3 = 1$: @ $(0.645)^k$ For $l_3 = 2$: Damping of $(0.725)^k$	Trough value: - 2 for $l_3 = 1$, -8 for $l_3 = 2$ But has zero offset	Entirely in (-)ve Region, (*high-stock-out risk*)	Fairly high for $l_3 = 1$: -14 *High* for $l_3 = 2$: -39 *Low responsiveness*	*Recovery Time: 16 for lag =1, 35 for lag = 2; High Design Intractability beyond lag of 2 periods*	-- do --
4	Moving Average control MA(ID)	Centre-line (-)ve	*High damping* , rate *can be set to any desired value* of $(\lvert K_3^1\rvert)^{1/(l_3+2)}$ Or $\sqrt[l+2]{\lvert K_3^1\rvert}$	Trough value and *(-)ve Offset* of -7.471 for $l_3 = 5$	Entirely in (-)ve region (*High stock-out risk*)	Infinite. Complete *loss of responsiveness*	*Recovery Time is infinite; poor Recovery; But Lags of any high value can be handled. High Design Tractability for any high value of lag*	*Fluctuations quickly damped out But system is left with a high (-)ve Offset; Poor Recovery,; but High Design Tractability for arbitrary large lags*
5	MA(ID) with recovery flow	(-)ve initially, but can be *pulled up optimally to above zero*	*High damping, rate can be set to any desired value* of $(\lvert K_3^1\rvert)^{1/(l_3+2)}$Or $\sqrt[l+2]{\lvert K_3^1\rvert}$	Low trough value of -7.471 for $l_3 = 5$, But Offset *rapidly/ optimally pulled up to above zero*	Respons e rapidly pulled up to above zero. *Low stock-out risk*	Low, 8 for lag = 5 *High degree of Responsiveness*	*Recovery Time = 10 even for high Lag =5; Lags of any high value can be handled. High Design Tractability for any high value of lag*	*Fluctuations quickly damped out System restored to normalcy rapidly. Rapid Recovery; High design Tractability for arbitrarily large lags*

Table 2 Performance Summary and Operational Significance

Sl. No	Type of control	Average inventory levels, And Permanent inventory levels	Base stock Requirement (based on Trough value)	Stock-out Risk (time in (-)ve inv.)	Rate of settling down (Damping Rate)	Responsiveness (inverse relation with Lag-specific backorders)	Special Characteristics/ Special Caution	Recovery and Design Tractability
1	Proportional to inventory	Very low and below normal Operation levels (high Offset)	high	High	Near perpetual fluctuations, Settles down very slowly	Very low, Complete loss of responsiveness	Long lasting high amplitude oscillations, System is never restored to its original state, Trade-off between damping vs. base stock requirements and stock-out risk	*Very Slow Recovery; High Design Intractability beyond lag of 1 period*
2	Proportional-integral	Low initially but pulled up to zero slowly	High (7 units) for lag of 2 periods	High to moderate	slow	low	Fluctuations damped out and system restored to its original state, but slowly	*Slow Recovery; High Design Intractability beyond lag of 2 periods*
3	Proportional-integral-derivative	Low initially but pulled up to zero slowly	High (7 units) for lags of 2 periods	moderate	slow	moderate	Same as for PI control (Sl. No 2 above)	Same as for PI control (Sl. No 2 above)
4	MA(ID) control	Very low	Very high	high	rapid	Very low	Fluctuations damped out rapidly, but	*Slow Recovery; but*

							system not restored to original state, leaving inv level depleted	*High Design Tractability for any high value of lag*
5	MA(ID) with recovery flow	Low initially , but pulled up rapidly to zero	Low even for large lag of 5 periods	moderate	rapid	High	Fluctuations damped out rapidly, system restored to normalcy rapidly/optimally	*Rapid Recovery AND High Design Tractability for any high value of lag*

From the tables above, we can see that the MA(ID) control-with recovery flow shows the best performance. It shows a high degree of responsiveness and rapid recovery, as well as exhibits good design tractability even for arbitrarily large lags. The control scheme hence meets both our design stipulations, and hence could be the preferred form of replenishment control for a SC operating under large replenishment lags and uncertain demand conditions.

In the SC, since the input flows into any unit in the chain would be the 'demand' (flows) for the preceding upstream units in the chain, *maintaining stable inventory and replenishment flow levels at the warehouse would be fundamental to ensuring stable operation in the upstream units* of the chain. To this end, the MA(ID) control with recovery flow attempts to restore normalcy in a more uniform manner. This control scheme hence could be of good advantage to the SC operators.

9. The Effect of Lag on System Performance and Discussion

We now wish to examine specifically the effect of lag on system performance. We do this by comparing the performance of our system with two other closely related systems studied in the literature as under:

a) One with the *replenishment lag acting in isolation (a sudden increase in lag but in the absence of any demand disturbance*. In such a situation, the disturbance would be caused entirely by the sudden increase in replenishment lag. Such a system has been modelled by taking the demand disturbance as zero, but making the lag in the system non-zero from the first period onwards (sudden increase in lag).

b) One with the same *step demand disturbance acting in isolation*, i.e one with a demand disturbance as in our system, but *with zero replenishment lag*.

The effect of these differences in the assumptions can be substantial as they affect the Initial Conditions (ICs) of the system equations, which in turn determine the magnitude of the system response, which can then become quite different.

We show the comparison between them in Table 3 below.

Table 3 Comparative Performance of different controls on the three types of systems

Sl No	Control Type	Performance Metric	System 1: *Sudden Increase in Lag* with *Zero Demand Disturbance* [31]	System 2: P*re-existing Lags* with *Sudden Increase in Demand* (This study)	System 3: *Zero Lag* with *Sudden Increase in Demand* [32]
1	P(I)	Trough Value/Offset	-2 for lag =1	-1.65 to -6.75 for lag = 1	-2
		Amplitude of Oscillations	1.1 units, perpetual oscillations	up to 1.8 units, perpetual oscillations	1.0 units perpetual oscillations
		Damping	Max possible: $O(2/3)^k$, low	Max possible: $O(2/3)^k$, low	Max: $O(1/2)^k$, moderate
		Stock-out Risk	Oscillatory	Oscillatory, but highest due to highest amplitude fluctuations	Oscillatory, but lowest due to lowest amplitude
		Max Back-order posn./Lag-specific Back-orders	Oscillatory between -4 and 0	Infinite, loss of Responsiveness	Oscillatory between -2 and 0
		Inv. Variance	$6.05\sigma^2$ low	$6.05\sigma^2$ low	for $K_3 = -1$: unbdd $3.74\sigma^2$ for $K_3 = -1/4$
		Design Tractability for High Lag	Low beyond lag of one period	Low beyond lag of one period	Not Applicable
		Stability Region	For lag = 1: $-0.619 \le K_3 < 0$	For lag = 1: $-0.619 \le K_3 < 0$	For zero lag: $-1 \le K_3 < 0$

		Lag-adjusted Recovery Time	23 for Lag = 1 (Slow Recovery) *Intractable for higher Lags*	Infinite (Very Slow Recovery)	* Infinite (Very Slow Recovery) But Not Comparable
2	PI(I)	Trough Value	-3 for lag = 1 -4 for lag = 2	-4 for lag = 1 -6.85 for lag = 2	- 2 for zero lag and unit step demand
		Damping	$O(0.79)^k$ for lag = 1 $O(0.94)^k$ for lag = 2	Same as for zero input case	$O(2/3)^k$ for zero lag and unit step demand
		Stock-out Risk	Moderately high for lag = 1 very high for lag =2	Highest both for lag = 1 and 2	(High)
		Max. Back-order posn./ Lag-specific Back-orders	-5 for lag = 1 low -126/-63 for lag = 2	-30 for lag = 1 High -139/-70 for lag = 2 Very High	(- 11 for zero lag and unit step demand)
		Inv. Variance	$3.93\sigma^2$ for lag =1 $181\sigma^2$ for lag = 2	Same as for zero input case	$2.5\sigma^2$ for lag =0 and unit step demand
		Design Tractability for High Lag	Low for lag beyond 2 periods	Same as for zero input case	Not Applicable
		Lag-adjusted Recovery Time	32 for Lag = 1 50 for Lag =2 (Slow Recovery)	25 for lag = 1 33 for Lag =2 (Slow Recovery)	* 23 for Lag = 0 (Slow Recovery)
3	PID(I)	Trough Value	-1 for lag = 1 -3 for lag = 2	-2 for lag = 1 -8 for lag = 2	-2.5 for zero lag and step demand
		Damping	$O(0.645)^k$,lag = 1 $O(0.725)^k$,lag = 2	Same as for zero input case	$O(0.5)^k$ for zero lag and unit step demand
		S/O Risk	High (entirely in (-)ve region	High (entirely in (-)ve region	High (entirely in (-)ve region
		Max Back-order posn./ Lag-specific Back-order	-4 for lag = 1 Low -47/-24 for lag = 2 High	-14 for lag = 1 High -116/-78 for lag = 2 Very High	(-15 for zero lag and unit step demand)
		Inv. Variance	$4.16\sigma^2$ for lag =1 $15.6\sigma^2$ for lag = 2	Same as for zero input case	$2.5\sigma^2$ for zero lag and unit step demand
		Design Tractability for High Lag	Low for lag beyond 2 periods	Low for lags beyond 2 periods	Not Applicable
		Lag-adjusted Recovery Time	22 for Lag = 1 37 for Lag =2 (Slow Recovery)	16 for Lag =1 33 for Lag =2 (Slow Recovery)	* 12 for Lag = 0 (Moderately rapid Recovery)
4	MA(ID)	Trough/Offset	-5.33 for lag = 5	-7.47 for lag =5	-2 for MA order = 3
		Damping	High, $O((\lvert K_3^1\rvert^{1/(l_3+3)})^k)$, can be set to any desired value	Same as for zero input case	High $O(1/(r+1))^k$ $O(1/4)^k$ for MA order = 3
		S/O Risk	High (entirely in (-)ve region	High (entirely in (-)ve region	Low, inventory pulled up rapidly
		Max. B/Orders /Lag-Specific B/O	Infinite, loss of responsiveness	Same as for zero input case	-4 very low, and hence very High Responsiveness
		Inv. Variance	Low, $20.54\sigma^2$ even for lag = 5	Same as for zero input case	* $10\sigma^2$ for r = 3 (computed in Appendix)
		Design Tractability for High Lag	High for any arbitrary high lag	High for any arbitrary high lag	Not Applicable
		Lag-adjusted Recovery Time	Infinite	Infinite	* 5 for Lag =0 (Rapid Recovery)
5	MA(ID)	Trough/Offset	-5 for lag = 5	-7.47 for lag = 5	*** Not Available in the Literature**
	with recovery flow	Damping	High, $O((\lvert K_3^1\rvert^{1/(l_3+3)})^k)$, can be set to any desired value	High, $O((\lvert K_3^1\rvert^{1/(l_3+3)})^k)$, can be set to any desired value	*Will be the same as for MA(ID) control
		S/O Risk	Low, Inv. pulled up rapidly	Low, Inv. Pulled up rapidly	*Will be low
		Max. B/O/ Lag-Specific B/O	Low, -19.18/-4, even for lag = 5	Moderate, -40/-8 even for lag =5	*Will be low
		Inv Variance	Low, $20.54\sigma^2$ even for lag = 5	Same as for zero input case	*Will be the same as for MA(ID) control

		Design Tractability for High Lag	High for any arbitrary high lag	High for any arbitrary high lag	Not Applicable
		Lag-adjusted Recovery Time	5 even for High Lag = 5 (Rapid Recovery)	5 even for High Lag = 5 (Rapid Recovery)	* Will be low (Rapid Recovery)

(***Note1**: The *starred entries for System 3 for Control No 5 are not reported in the literature, and have been arrived at by us from the reported results for system 3, using the logic of adding the recovery flow).

Note 2: To ensure a valid comparison, we have taken system 1 with *unit nominal flow profile*, *whose effects, under replenishment lags would be similar in magnitude to that of a unit step increase in demand without lags,* thereby ensuring the *numerical, order-of-magnitude*, and *logical validity* of our comparison.

The significant points that we can note from the comparisons are the following:

1) Firstly, comparing all three systems, we can note:

a) All the performance metrics that are based on the LHS Operator term in the system LDE remain unchanged in all three cases, as can be expected; this is because the *change in input and/or change in ICs, does not in any way affect the LHS Operator, which is a function of the type of control only*. Hence, the following metrics remain unchanged: *Damping rate, Inventory Variance, and Design Tractability*.

b) The other performance characteristics viz. *Trough value*, *Offset, Stock-out risk,* and *Back-order position (Responsiveness), and Recovery time* are affected by the ICs and demand disturbance (RHS of the LDE).

2) We now compare the effect of pre-existing lags in the system (System 2 which is the one studied in our paper) with that of zero lag (System 3) for the *same demand disturbanc*e.

Comparing Systems 2 and 3, we can note the *deterioration* in nearly all the performance metrics *with the presence of lag* in the system (for system 2, our study), with the *extent of deterioration again increasing substantially with increase in lag*. In particular, both the back-order position as well as *Recovery Time deteriorate rapidly with increase in lag, leading to poor Responsiveness and Recovery.* While this is certainly to be expected, the results above serve to measure and quantify the *extent of deterioration* with increase in lag.

Moreover, for a responsive chain it is these latter performance metrics in (b) above that would be of greater importance, hence pointing to a *substantial deterioration* in *Responsiveness and Recovery performance* of the chain *with increase in lags*.

3) Lastly, comparing systems 1 and 2: The *Responsiveness* and *Recovery* performance of system 2 (pre-existing lag with demand disturbance which is our system) is poorer for all types of controls than that of system 1(with no demand disturbance but a sudden increase in lag), thereby highlighting the increased difficulty in controlling the system in the presence of *even pre-existing* lags when also subject to demand disturbances (even if lags are pre-existing in the system).

10. Managerial Implications

The managerial implications of the results above are quite clear.

First, if a SC *operating under large replenishment lags* and following a responsiveness strategy wishes to build-in a high degree of responsiveness and recovery capability in the face of sudden demand disturbances, it could benefit by adopting such dynamic controls, in particular the MA(ID) type of ordering controls with recovery flows.

Second, if the system has large replenishment lags, it would be more difficult to control (as has been shown through the comparative analysis above) and hence would require closer monitoring and control. And it is precisely in such circumstances that the dynamic controls derived herein would automatically build-in a good degree of responsiveness and recovery capability in the system. Such systems would restore normalcy rapidly and automatically, and external human intervention could be limited to extreme cases.

Third, and from the point of view of practice, these controls can be easily built into the Warehouse Management System software and can bring in a good degree of *in-built responsiveness and recovery capability* in automated warehouse operations.

11. Conclusion

Warehouse replenishment management systems can be put to a severe test by the presence of large replenishment lags in the system when operating in uncertain demand environments. Such lags serve to magnify the disruptive effects of demand disturbances on the SC, resulting in poor responsiveness and slow recovery. Such systems could benefit by the use of dynamic controls as studied herein. Among the dynamic controls, the conventional P, PI, and PID controls show poor performance, exhibiting low damping, high stock-out risk, low responsiveness and slow recovery. More importantly, they exhibit design intractability for lags beyond two periods. The MA(ID) control performs better with a high rate of damping but leaves the system with a large negative offset. Wherea, the MA(ID) control with recovery flow is able to pull up the inventory levels very quickly and thereby reduces both stock-out risks, as well as back-orders and recovery time, and thus exhibits a high degree of responsiveness and recovery. And most importantly, it exhibits a high degree of design tractability even for arbitrarily large lags.

Thus, when a responsive SC with large replenishment lags, operating in an environment of uncertain demand wishes to build-in good responsiveness and recovery capability, it could benefit from the use of such dynamic controls.
Further the comparative results in the paper highlight and underscore the greater difficulty in controlling the system in the presence of even pre-existing lags, and simultaneously also reinforce the utility of dynamic controls in such situations.
The results derived herein have direct applications in warehouse management systems on the one hand and would also be of relevance in the study of SC Resilience on the other.
A limitation of this study is that composite controls have not been studied herein, wherein one type of control is composed with another type, to exploit the advantages of each of the constituent controls in the composition. These point to directions for further work which could take up these and other such cases, which could further enhance the responsiveness and recovery performance of such SCs.
Other approaches could be through empirical studies using such controls in field settings as also computational studies.

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Appendix

Inventory Variance Computation for MA(ID) Control for Zero-lag and unit step demand (for the entry for System 3 in comparison Table 3 in section 9):

The stochastic LDE is ([22]):

$[1 - L/(r+1)]^{r+1} x_3(k)^{stoc} = -K_3^0 \varepsilon(k-2) - \varepsilon(k)$

Where the terms on the RHS are for the demand trigger term as well as the demand term given by : $-K_3^0 r_3(k-1) - r_3(k+1)$ in the original LDE.

For r = 3 : we have

$[1 - (L/4)]^4 x_3(k)^{stoc} = -K_3^0 \varepsilon(k-2) - \varepsilon(k) = [1 - L + (3/8)L^2 - (1/16)L^3 + (1/256)L^4] x_3(k)^{stoc}$

Since the LHS Operator is stable, using the infinite MA representation for the stochastic component ([15]), yields:

LHS Opr Term	Coefft of $\varepsilon(k)$	Coefft of $\varepsilon(k-1)$	Coefft of $\varepsilon(k-2)$	Coefft of $\varepsilon(k-3)$	Coefft of $\varepsilon(k-4)$	Coefft of $\varepsilon(k-5)$	Coefft of $\varepsilon(k)$
1	β_0	β_1	β_2	β_3	β_4	β_5	β_k
- L		$-\beta_0$	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$	$-\beta_{k-1}$
$+(3/8)L^2$			$(3/8)\beta_0$	$(3/8)\beta_1$	$(3/8)\beta_2$	$(3/8)\beta_3$	$(3/8)\beta_{k-2}$
$-(1/16)L^3$				$-(1/16)\beta_0$	$-(1/16)\beta_1$	$-(1/16)\beta_2$	$-(1/16)\beta_{k-3}$
$+(1/256)L^4$					$-\frac{1}{256}\beta_0$	$-\frac{1}{256}\beta_1$	$-\frac{1}{256}\beta_{k-4}$
RHS	- 1	0	$-K_3^0$	0	0	0	0

Which yields: $\beta_0 = -1 = \beta_1, \beta_2 = -5/8 - K_3^0, \beta_3 = -5/16 - K_3^0$, and $\beta_k - \beta_{k-1} + (3/8)\beta_{k-2} - (1/16)\beta_{k-3} + (1/256)\beta_{k-4} = 0, k \geq 4$, which has the solution as:

$\beta_k = (C_0 + C_1 k + C_2 k^2 + C_3 k^3)(1/4)^k$ with the ICs as above. This yields the system of equations:

$$\begin{pmatrix} \beta_0 \\ 4\beta_1 \\ 16\beta_2 \\ 64\beta_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -10 - 16K_3^0 \\ -20 - 64K_3^0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -34 \\ -116 \end{pmatrix}, \text{ which yields: } \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2.667 \\ -1 \\ -4.1667 \end{pmatrix}$$

Where the last equality is by substituting for $K_3^0 = 1.5$.

Hence the limiting inventory variance is given by: $lim_{k\to\infty} var(x_3(k)) = \sum_{k=0}^{\infty} \beta_k^2 \sigma^2 = \sigma^2(1 + 1 + (-2.125)^2 + (-1.813)^2 + \sum_{k=4}^{\infty} \beta_k^2)$, where $\beta_k = (-1 - 2.667k - k^2 - 4.167k^3)(1/4)^k, \quad k \geq 4.$

The square of these terms is dominated by the last term: $4.167^2 k^6 (1/16)^k$, which has a maximum at k = 3. Hence, we have, $\sum_{k=4}^{\infty} \beta_k^2 \leq (17.36)3^6(1/16)^3(1/16)\sum_{l=0}^{\infty}(1/16)^l = (0.193)(16/15) = 0.21$, and hence we have:

$lim_{k\to\infty} var(x_3(k)) =$
$= \sigma^2(1 + 1 + (-2.125)^2 + (-1.813)^2 + 0.21) = 10.01\sigma^2$.