Joint Pricing and Inventory Control for Non-instantaneous Deteriorating Items with Stochastic Demand

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***Abstract*—In recent years inventory and pricing of deteriorating items has gained an enormous attention by many researchers. In this study, an inventory system for non-instantaneous deteriorating items with stochastic demand is modeled. This model has the assumptions that shortages are allowed and backlogging rate is variable where the last one is defined as a function of waiting time for the next replenishment. The objective is to maximize the total profit per unit time by finding the optimal selling price and replenishment schedule simultaneously. The concavity of the function is proved with a unique optimal solution. Thereby we provide an algorithm for finding the optimal solution. Finally, the authors present a numerical example to illustrate the theoretical results. A sensitivity analysis for the optimal solution with respect to major parameters is also carried out.**

***Keywords*—** *pricing, inventory control, non-instantaneous deteriorating items, and stochastic demand*

1. Introduction

The main costs of an inventory system are known as the cost of purchasing and holding for the enterprises. Enterprises always are looking for different methods to reduce inventory costs whereas customer satisfaction is increased. To obtain this purpose, different models can be formulated. Economic Order Quality (EOQ) model is one of the main models which have a flexibility to be extended to overcome more realistic situations. Moreover, deterioration assumption is one of the most important issues that is considered by many researchers to modify EOQ model. Ghare and Schrader (1963) introduced constant deterioration rate for items in their models [1]. Thereafter, Philip (1974) presented the inventory model with three-parameter Weibull distribution rate [2]. Finally, Goyal and Giri (2001) provided an excellent and perfect review of deteriorating inventory literatures [3].

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The shortage of the inventory is another important issue that may occur in the inventory system which means that some customers but not all of them might wait until the next replenishment happens or as mentioned in literature we encountered with the partial backlogging. Abad (1996, 2001) investigated the model with assumption that partial backlogging is allowable [4], [5]. In his model the partial backlogging rate depends on waiting time for next replenishment. Another works that have been investigated partial backlogging are due to Dye (2007) and San Jose et al. (2006) [6], [7].

Finally, in the real world the price is one of the main factors that affect the demand. Generally, any decrease in the selling price by the enterprises lead to increase in demand by the customers. But decrease in price cannot be done arbitrary and is mostly restricted by total enterprises cost. Therefore, choosing an appropriate price strategy is one of the important criteria to maximize enterprises profit. In recent years, many experts are looking to find methods which consider inventory level and pricing for deterioration items simultaneously which leads to maximize the enterprises profit. Eilon and Mallaya (1966) firstly investigated inventory model with price-dependent demand [8]. Afterwards, Chen et al. (2006) considered pricing and inventory control model for deterioration items in regard to price and time-dependent demand with shortage as a partial backlogging [9]. In their work, period times were finite for n periods. Dye et al. (2007)

developed an inventory and pricing strategy for deteriorating items with shortages [10]. Demand and deterioration rate are continuous and differentiable function of price and time, respectively.

A recent developed issue which affected pricing and inventory control models is non-instantaneous deterioration phenomena. For the first time, Wu et al. (2006) introduced non-instantaneous deterioration phenomena[11]. They proposed inventory model for non-instantaneous deterioration items while demand depends on inventory level. In the real world non-instantaneous deterioration exist for first-hand vegetables and fruits that they have a short span of maintaining fresh quality, in which there is almost no spoilage. Computations show that if deterioration of items does not commence after entering to the system, then instantaneous deterioration models may lead to incorrect replenishment policy. Ouyang et al. (2006) presented an inventory model for non- instantaneous deteriorating items with permissible delay in payments. But their model does not included the concavity prove [12]. Chung (2009) considered this issue and presented complete proofs for Ouyang et al. (2006) model [13]. Valliathal and Uthayakumar (2011) investigated pricing and replenishment policy for non-instantaneous deterioration items with constant deterioration rate, time and price demand [14]. Maihami and Nakhai (2012) developed a pricing and inventory control model for non-instantaneous deteriorating items. They consider the demand as a linearly decreasing function of the price and a decrease (increase) exponentially function of time [15].

In this study, to obtain more realistic condition, an inventory model for non-instantaneous deterioration items with stochastic demand has been developed. It is inevitable to consider the demand as stochastic to solve uncertainly for tackling the problem. Shortages are allowed and backlogging rate is variable which is defined as a function of waiting time for the next replenishment. Costs are including purchase cost, inventory holding cost, shortage cost due to backlogging, opportunity cost due to lost sales, ordering cost and deterioration cost. Our objective is to determine the optimal selling price and replenishment schedule simultaneously such that the expected total profit per unit time is maximized.

Following this, in Section 2, assumptions and notations used throughout this paper are given. In section 3, the mathematical model is presented .In Section 4, we establish the necessary conditions for finding an optimal solution. For any given selling price, we then establish conditions for the optimal solution to exist and also be unique. Moreover, the expected total profit is proved that a concave function of the selling price. Next, in Section 5, we present a simple algorithm for finding the optimal selling price and inventory control variables. In Section 6, a numerical example is given and, finally, we provide a summary and some suggestions for future work in Section 7.

2. Notation and Assumptions

The following notation and assumptions are used throughout the paper.

Notation.

k: The ordering cost per order
c: The purchasing cost per unit
h: The holding cost per unit per unit time
s: The shortage cost per unit per unit time
o: The unit cost of lost sales
p: The selling price per unit, where $p > C$
ε: stochastic part of demand, independent of price
μ: the mean of ε
E : expectation
θ: The parameter of the deterioration rate
t_d: The length of time in which the product exhibits no deterioration
t_1: The length of time in which there is no inventory shortage
T: The length of the replenishment cycle time
Q: The order quantity
p:* The optimal selling price per unit
t^*_1: The optimal length of time in which there is no inventory shortage
T:* The optimal length of the replenishment cycle time
Q:* The optimal order quantity
I_1 (t): The inventory level at time $t \in [0, t_d]$
I_2 (t): The inventory level at time $t \in [t_d, t_1]$, where $t_1 > t_d$
1_3(t): The inventory level at time $t \in [t_1, T]$
I_0: The maximum inventory level
S: The maximum amount of demand backlogged
TP (p, t_1, T): The expected total profit per unit time of the inventory system
TP*: The expected optimal total profit per unit time of the inventory system, that is, $TP^* = TP(p^*, t^*_1, T^*)$.

Assumptions:

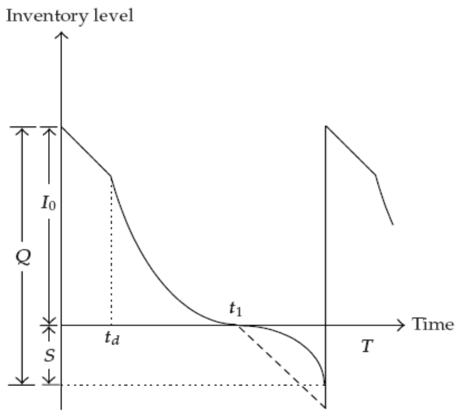
1. A single non-instantaneous deterioration item is modeled.The replenishment rate is infinite and the lead time is zero.
2. The demand rrate D(p) is a non-negative, continuous, decreasing and concave function of the selling price p, that is, D' (p) < 0 and D" (p) < 0 i.e., its hessian is negative definite.
3. The additive form of D(p) and ε is considered as follows

$$d(p, \varepsilon) = D(p) + \varepsilon$$

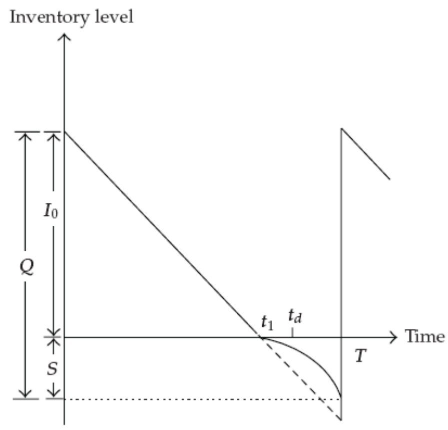
4. The distribution of ε is continuous
5. During the fixed period, t_d, there is no deterioration and at the end of this period, the inventory item deteriorates at the constant rate θ.
6. There is no replacement or repair for deteriorated items during the period under consideration.
7. Shortages are allowed to occur. It is assumed that only a fraction of demand is backlogged. Furthermore the longer the waiting time, the smaller the backlogging rate. Let B(x) denote the backlogging rate given by $B(x) = 1/(1 + \delta x)$, where *x* is the waiting time until the next replenishment and δ>0 is a positive backlogging parameter. We use the notation used in Abad (1996) [4].

3. Model Formulation

In this research the replenishment problem of a single non-instantaneous deteriorating item with partial backlogging is considered. The inventory system is as follows. I_0 units of item arrive at the inventory system at the beginning of each cycle and decrease to zero due to the demand and deterioration. Shortage may occur during the current order cycle. Here, demand has a stochastic distribution which depends on selling price. Based on the values of t_1 and t_d there are two possible scenarios: (1) $t_1 \geq t_d$ and (2) $t_1 \leq t_d$ (see Fig.1). These scenarios are discussed as follows



Scenario 1: $t_1 \geq t_d$



Scenario 2: $t_1 \leq t_d$

Figure 1. Graphical representation of inventory system

Scenario 1 ($t_1 \geq t_d$). In this scenario, the deterioration of items happens prior to the shortage point (t_1). Simply, the deterioration is not provided during the time interval $[0, t_d]$, that is in the interval $[0, t_d]$ the inventory level only decreases due to demand. Subsequently during the time interval $[t_d, t_1]$ the inventory level decrease to zero due to both demand and deterioration. Finally, a shortage occurs due to both demand and partial backlogging during the time interval $[t_1, T]$. The whole process is repeated for the next period.

As described before, during the time interval $[0, t_d]$, the inventory level decreases due to demand only. Hence the differential equation representing the inventory status is given by

$$\frac{dI_1(t)}{dt} = -D(p) - \varepsilon \qquad 0 \le t \le t_d \tag{3.1}$$

With the boundary condition $I_1(0) = I_0$. By solving above equation, it yields.

$$I_1(t) = -(D(p) + \varepsilon)t + I_0 \qquad 0 \le t \le t_d \tag{3.2}$$

The inventory decreases due to the combined effects of the demand and deterioration in the interval $[t_d, t_1]$. Thus, the differential equation representing the inventory status is given by

$$\frac{dI_2(t)}{dt} = -D(p) - \varepsilon - \theta I_2(t) \qquad t_d \le t \le t_1 \tag{3.3}$$

With the boundary condition $I_2(t_1) = 0$. By solving above equation, it yields.

$$I_2(t) = \frac{(D(p)+\varepsilon)}{\theta}\left[e^{\theta(t_1-t)} - 1\right] \qquad t_d \le t \le t_1 \tag{3.4}$$

From Fig. 1 we have $I_1(t)$ and $I_2(t)$ at point $t = t_d$ are equal ($I_1(t_d) = I_2(t_d)$). Thus the maximum inventory level for each cycle can be obtained by

$$I_0 = \frac{(D(p)+\varepsilon)}{\theta}\left[e^{\theta(t_1-t_d)} - 1\right] + (D(p)+\varepsilon)t_d \tag{3.5}$$

Substituting (3.5) into (3.2) gives

$$I_1(t) = \frac{(D(p)+\varepsilon)}{\theta}\left[e^{\theta(t_1-t_d)} - 1\right] + (D(p)+\varepsilon)(t_d - t) \qquad 0 \le t \le t_d \tag{3.6}$$

During the interval $[t_1, T]$, the demand at time t is partially backlogged according to the fraction B (T - t). Thus, the inventory level at time t is governed by the following differential equation:

$$\frac{dI_3(t)}{dt} = -\frac{D(p)+\varepsilon}{1+\delta(T-t)} \qquad t_1 \le t \le T \tag{3.7}$$

With condition $1_3(t_1) = 0$. The solution of (3.7) is

$$I_3(t) = -\left(\frac{D(p)+\varepsilon}{\delta}\right)\{\ln[1+\delta(T-t_1)] - \ln[\ 1+\delta(T-t)]\} \qquad t_1 \le t \le T \tag{3.8}$$

Putting t = T into (3.8), the maximum amount of demand backlogged per cycle is obtained as follows

$$-\int_{t_1}^{T} I_3(t)\,d(t) = \frac{(d(p)+\varepsilon)[\delta(T-t_1) - \ln(1+\delta(T-t_1)\ ]}{\delta^2} \tag{3.9}$$

from (3.5) and (3.9), the order quantity per cycle is:

$$Q = I_0 - I_3(T) = \frac{(D(p)+\varepsilon)}{\theta}\left[e^{\theta(t_1-t_d)} - 1\right] + (D(p)+\varepsilon)t_d + \left(\frac{D(p)+\varepsilon}{\delta}\right)\ln[1+\delta(T-t_1)] \tag{3.10}$$

$$E(Q) = \frac{(D(p)+\mu)}{\theta}\left[e^{\theta(t_1-t_d)} - 1\right] + (D(p)+\mu)t_d + \left(\frac{D(p)+\mu}{\delta}\right)\ln[1+\delta(T-t_1)] \tag{3.11}$$

Next, the costs and revenue of the system can be presented by following seven items per cycle

1) The ordering cost is *k.*

2) The expected inventory holding cost (H C) is

$$Hc = h\left(\int_0^{t_d} E(I_1(t))dt + \int_{t_d}^{t_1} E(I_2(t))dt\right)$$

$$= h(D(p)+\mu)\{\frac{t_d}{\theta}\left[e^{\theta(t_1-t_d)} - 1\right] + \frac{t_d^2}{2} + \frac{1}{\theta^2}\left[e^{\theta(t_1-t_d)} - \theta(t_1-t_d) - 1\right]$$

3) The expected shortage cost due to backlog (Sc)

$$Sc = s\int_{t_1}^{T} -E(I_3(t))dt = s\,\frac{(D(p)+\mu)\left[\delta(T-t_1) - Ln\left(1+\delta(T-t_1)\right)\right]}{\delta^2}$$

4) The expected opportunity cost due to lost sales (OC) is

$$Oc = o\int_{t_1}^{T}(D(p)+\mu)\left(1-\beta(x)\right)dx = o\left\{\frac{(D(p)+\mu)\left[\delta(T-t_1) - Ln\left(1+\delta(T-t_1)\right)\right]}{\delta}\right\}$$

5) The expected purchase cost (PC) is

$$Pc = E(cQ) = c\frac{(D(p)+\mu)}{\theta}\{\left[e^{\theta(t_1-t_d)} - 1\right] + t_d + \left(\frac{1}{\delta}\right)\ln[1+\delta(T-t_1)]\}$$

6) The expected deterioration cost (Mc) is

$$Mc = m\left(\int_{t_d}^{t_1} \theta E(I_2(t))dt\right) = m(D(p)+\mu)\left\{\frac{1}{\theta}\left[e^{\theta(t_1-t_d)} - \theta(t_1-t_d) - 1\right]\right\}$$

(7) The expected sales revenue (*Sr)* is

$$Sr = p\left(\int_0^{t_1} E(D(p,\varepsilon))dt - E(I_3(T))\right) = p\,\{(D(p)+\mu)(t_1 + Ln(1+\delta(T-t_1))/\delta)\}$$

Finally, the expected total profit per unit time of scenario 1 (TP_1 (p, t_1, T)) is given by:

$$TP_1(p,t_1,T) = \frac{1}{T}\{Sr - Pc - Oc - Hc - Sc - Mc - K\}$$

$$= \frac{D(p)+\mu}{T}\left\{\left(p - c + \frac{s+\delta o}{\delta}\right)\left[t_1 + \frac{\ln[1+\delta(T-t_1)]}{\delta}\right] - \frac{\theta(c+ht_d+m)+h}{\theta^2} \times \left[e^{\theta(t_1-t_d)} - \theta(t_1-t_d) - 1\right]\right.$$

$$\left. - ht_d t_1 + \frac{ht_d^2}{2} - \frac{s+\delta o}{\delta}T - \frac{k}{D(p)+\mu}\right\} \qquad (3.12)$$

Scenario 2 ($t_1 \leq t_d$). In this scenario, the deterioration happens after shortage point, or simply there is no deterioration at all. This implies that the optimal replenishment policy for the enterprises is to sell out all stock before the deadline at which the items start to decay. Under these conditions, the model becomes the traditional inventory model with a shortage. By using similar approach as in scenario 1, the expected order quantity per order, Q, and the expected total profit per unit time (TP_2 (p, t_1, T)) can be obtained as follows:

$$E(Q) = (D(p)+\mu)t_1 + \frac{(D(p)+\mu)}{\delta}\ln[1+\delta(T-t_1)] \qquad (3.13)$$

$$TP_2(p,t_1,T) = \frac{1}{T}\{Sr - Pc - Oc - Hc - Sc - Mc - K\}$$

$$= \frac{D(p)+\mu}{T}\left\{(p-c)\left[t_1 + \frac{\ln[1+\delta(T-t_1)]}{\delta}\right] - \frac{ht_1^2}{2} - \frac{s+\delta o}{\delta}\left[T - t_1 - \frac{\ln[1+\delta(T-t_1)]}{\delta}\right]\right.$$

$$\left. - \frac{k}{D(p)+\mu}\right\} \qquad (3.14)$$

Summarizing the above discussion, the expected total profit per unit time of the inventory system is as follows:

$$TP(p,t_1,T) = \begin{cases} TP_1(p,t_1,T), & \text{if } t_1 \geq t_d \\ TP_2(p,t_1,T), & \text{if } t_1 \leq t_d \end{cases}$$

Where TP_1 (p, t_1, T) and TP_2 (p, t_1, T) are given by (3.12) and (3.14), respectively.

4. Theoretical Results

The objective is to determine the optimal selling price and replenishment schedule simultaneously such that the expected total profit per unit time is maximized. To achieve this goal, we should prove that for any given *p*, the optimal solution of (t_1, T) not only exists but also is unique. Next for any given value of (t_1, T), there exists a unique *p* that maximizes the expected total profit per unit time. The detailed solution procedures for two scenarios are as follows.

Scenario 1 ($t_1 \geq t_d$). First, for any given p, the necessary conditions for the expected total profit per unit time in (3.12) to be maximized are following equals simultaneously:

$$\frac{\partial TP_1(p,t_1,T)}{\partial t_1} = \frac{D(p)+\mu}{T}\left\{\left(p - c + \frac{s+\delta o}{\delta}\right)\left[\frac{\delta(T-t_1)}{1+\delta(T-t_1)}\right] - \frac{\theta(c+ht_d+m)+h}{\theta} \times \left[e^{\theta(t_1-t_d)} - 1\right] - ht_d\right\} = 0$$

$$\frac{\partial TP_1(p,t_1,T)}{\partial T} = \frac{D(p)+\mu}{T^2}\left\{\left(p - c + \frac{s+\delta o}{\delta}\right)\left[\frac{T}{1+\delta(T-t_1)} - t_1 - \frac{\ln[1+\delta(T-t_1)]}{\delta}\right] + \frac{\theta(c+ht_d+m)+h}{\theta^2} \times\right.$$

$$\left.\left[e^{\theta(t_1-t_d)} - \theta(t_1-t_d) - 1\right] + ht_d t_1 - \frac{ht_d^2}{2} + \frac{k}{D(p)+\mu}\right\} = 0$$

(4.1)

For notational convenience, let

$$M \equiv \frac{\theta(c+ht_d+m)+h}{\theta} > 0 \quad , \quad N \equiv \frac{s+\delta o}{\delta} > 0 \qquad (4.2)$$

Then, from (4.1), it can be found that

$$T = t_1 + \frac{M\left[e^{\theta(t_1-t_d)} - 1\right] + ht_d}{\delta\{p - c + N - M[e^{\theta(t_1-t_d)} - 1] - ht_d\}} \qquad (4.3)$$

$$\left\{(p-c+N)\left[\frac{T}{1+\delta(T-t_1)} - t_1 - \frac{\ln[1+\delta(T-t_1)]}{\delta}\right] + \frac{M}{\theta} \times \left[e^{\theta(t_1-t_d)} - \theta(t_1-t_d) - 1\right] + ht_d t_1 - \frac{ht_d^2}{2}\right.$$

$$\left. + \frac{k}{D(p)+\mu}\right\} = 0 \qquad (4.4)$$

respectively.

Due to *T* >t_1, from (4.3), it can be found that

$$t_1 < t_d + (1/\theta)\ln\left[\left(\frac{p - c + N + M - ht_d}{M}\right)\right. = t_1^b$$

Substituting (4.3) into (4.4) and simplifying gives

$$\left\{\{M[e^{\theta(t_1-t_d)} - 1] + ht_d\}\left(\frac{1}{\delta} - t_1\right) - \frac{(p-c+N)}{\delta} ln\left[\frac{(p-c+N)}{(p-c+N) - M[e^{\theta(t_1-t_d)} - 1] - ht_d}\right] + \frac{M}{\theta}\right.$$

$$\times \left[e^{\theta(t_1-t_d)} - \theta(t_1 - t_d) - 1\right] + ht_d t_1 - \frac{ht_d^2}{2} + \frac{k}{D(p)+\mu}\Big\} = 0 \quad (4.5)$$

Next, to find x ϵ (t_d, t_1^b) which satisfies (4.5), let

$$F(x) = \left\{M\left[e^{\theta(x-t_d)} - 1\right] + ht_d\right\}\left(\frac{1}{\delta} - x\right) - \frac{(p-c+N)}{\delta} ln\left[\frac{(p-c+N)}{(p-c+N)-M\left[e^{\theta(x-t_d)}-1\right]-ht_d}\right]$$

$$+\frac{M}{\theta} \times \left[e^{\theta(x-t_d)} - \theta(x - t_d) - 1\right] + ht_d x - \frac{ht_d^2}{2} + \frac{k}{D(p)+\mu}\Big\}$$

$x \in [t_d, t_1^b)$ (4.6)

Taking the first-order derivative of *F(x)* with respect to x ϵ (t_d, t_1^b), it is found that

$$\frac{F(x)}{dx} = -\theta M e^{\theta(x-t_d)}\left\{x + \frac{M\left[e^{\theta(x-t_d)} - 1\right] + ht_d}{\delta\{p - c + N - M\left[e^{\theta(x-t_d)} - 1\right] - ht_d\}}\right\} < 0 \quad (4.7)$$

Thus, F(x) is a strictly decreasing function in x ϵ [t_d, t_1^b). Furthermore, it can be shown that

Now let

$$G(p) \equiv F(t_d) = \frac{ht_d}{\delta} - \frac{p-c+N}{\delta} ln\left[\frac{p-c+N}{p-c+N-ht_d}\right] - \frac{ht_d^2}{2} + \frac{k}{D(p)+\mu} \quad (4.8)$$

This gives the following result.

Lemma 4.1. *For any given p,*

a. If G(p)≥ 0, then the solution of (t_1, T) which satisfies (4.1) *not only exists but also is unique,*

b. if G(p) <0, then the solution of (t_1, T) which satisfies (4.1) *does not exist.*

Proof: See Appendix A.

Lemma 4.2. *For any given p,*

a. *If G(p)≥ 0, then the expected total profit per unit time TP_1 (p,*t_1*, T) is concave and reaches its global maximum at the point (t_1, T) =* (t_{11}, *T_1), where (t_{11}, T_1) is the point which satisfies (4.1),*

b. *If G(p) < 0, then the total profit per unit time T* P_1 *(p,* t_1*, T) has a maximum value at the point* (t_1, T) = (t_{11}, T_1)*, where t_{11} = t_d and* $T_1 = t_d + h\,t_d/(\delta(p - C + N - h\,t_d))$.

Proof: See Appendix B.

The problem remaining in scenario 1 is to find the optimal value of *p* which maximizes *TP_1 (p, t_{11}, T_1).* Taking the first- and second-order derivatives of *TP_1 (p, t_{11}, T_1)* with respect to *p* gives

$$\frac{dTP_1(p,t_{11},T_1)}{dp} = \frac{D(p)'}{T_1}\Big\{(p - c + N)\left[t_{11} + \frac{ln[1 + \delta(T_1 - t_{11})]}{\delta}\right] - \frac{M}{\theta} \times \left[e^{\theta(t_{11}-t_d)} - \theta(t_{11} - t_d) - 1\right] - ht_d t_{11} + \frac{ht_d^2}{2} - NT_1)\Big\} + \frac{(D(p)+\mu)}{T_1}\left\{t_{11} + \frac{ln[1 + \delta(T_1 - t_{11})]}{\delta}\right\} \quad (4.9)$$

$$\frac{d^2TP_1(p,t_{11},T_1)}{dp^2} = \frac{D(p)''}{T_1}\Big\{(p - c + N)\left[t_{11} + \frac{ln[1+\delta(T_1-t_{11})]}{\delta}\right] - \frac{M}{\theta} \times \left[e^{\theta(t_{11}-t_d)} - \theta(t_{11} - t_d) - 1\right] - ht_d t_{11} + \frac{ht_d^2}{2} - NT_1)\Big\} + \frac{2D(p)'}{T_1}\left\{t_{11} + \frac{ln[1+\delta(T_1-t_{11})]}{\delta}\right\} < 0 \quad (4.10)$$

Where *D'(p)* and *D''(p)* are the first and second-order derivatives of *D (p)* with respect to *p*, respectively. By the assumptions *D'(p)* and *D''(p)* < 0, and it is known that the brace term in (4.10) is positive. Therefore $d^2TP_1\ (p,\ t_{11},\ T_1)/dp^2$ <0. Consequently, *TP_1 (p, t_{11}, T_1)* is a concave function of p for a given (t_{11}, T_1), and hence there exists a unique value of *p* (say *P_1)* which maximizes *TP_1 (p, t_{11}, T_1). P_1* can be obtained by solving *dTP_1 (p, t_{11}, T_1) /d p* = 0; that is, *P_1* can be determined by solving the following equation:

$$\frac{D(p)'}{T_1}\Big\{(p - c + N)\left[t_{11} + \frac{ln[1 + \delta(T_1 - t_{11})]}{\delta}\right] - \frac{M}{\theta} \times \left[e^{\theta(t_{11}-t_d)} - \theta(t_{11} - t_d) - 1\right] - ht_d t_{11} + \frac{ht_d^2}{2} - NT_1)\Big\} + \frac{(D(p)+\mu)}{T_1}\left\{t_{11} + \frac{ln[1 + \delta(T_1 - t_{11})]}{\delta}\right\} = 0 \quad (4.11)$$

***Scenario* 2 (t_1≤ t_d).** Similarly to scenario 1, for any given *p*, the necessary conditions for the expected total profit per unit time in (3.14) to be maximized are *∂TP_2 (p, t_1, T)/ ∂t_1* = 0 and *∂TP_2 (p,* t_1*, T)/ ∂T* = 0, simultaneously, which implies

$$(p - c + N)\frac{\delta(T-t_1)}{1+\delta(T-t_1)} - ht_1 = 0 \quad (4.12)$$

$$\frac{D(p)+\mu}{T^2}\left\{(p - c + N)\left[\frac{T}{1+\delta(T-t_1)} - t_1 - \frac{ln\,[1+\delta(T-t_1)]}{\delta}\right] + \frac{ht_1^2}{2} + \frac{k}{D(p)+\mu}\right\} = 0 \quad (4.13)$$

Respectively.

From (4.12), the following is obtained:

$$T = t_1 + \frac{ht_1}{\delta\{p - c + N - ht_1\}} \quad (4.14)$$

Substituting (4.14) into (4.13) gives

$$\frac{ht_1}{\delta} - \frac{(p-c+N)}{\delta} \ln\left[\frac{p-c+N}{p-c+N-ht_1}\right] - \frac{ht_1^2}{2} + \frac{k}{D(p)+\mu} = 0 \quad (4.17)$$

By using a similar approach as used in scenario 1, the following results are found.

Lemma 4.3. *For any given p,*

a. *If* G(*p*)≤0, *then the solution of* (t_1, *T*) *which satisfies* (4.12) *and* (4.13) *not only exists but also is unique,*
b. *If* G(*p*) > 0, *then the solution of* (t_1, *T*) *which satisfies* (4.12) *and* (4.13) *does not exist.*

Proof. The proof is similar to Appendix A, and hence is omitted here.

Lemma 4.4. *For any given p,*

a. *If* G(*p*)≤0, *then the total profit per unit time* $TP_2(p, t_1, T)$ *is concave and reaches its global maximum at the point* $(t_1, T) = (t_{12}, T_2)$*, where* (t_{12}, T_2) *is the point which satisfies (4.12) and (4.13),*
b. *If* G(*p*) > 0, *then the total profit per unit time* TP_2 (p, t_1, *T*) *has a maximum value a point* $(t_1, T) = (t_{12}, T_2)$*, where* $t_{12} = t_d$ *and* $T_2 = t_d + h\, t_d / (\delta(p - c + N - h\, t_d))$.

Proof. The proof is similar to Appendix B, and hence is omitted here.

Likewise, for a given (t_{12}, T_2), taking the first- and second-order derivatives of TP_2 (p, t_{12}, T_2) in (3.14) with respect to *p*, it is found that It can be shown that d^2TP_2 $(P, t_{12}, T_2)/dp^2 < 0$. Consequently, TP_2 (p, t_{12}, T_2) is a concave function of *P* for fixed (t_{12}, T_2), and hence there exists a unique value of *P* (say P_2) which maximizes TP_2 (p, t_{12}, T_2). P_2 can be obtained by solving dTP_2 $(p, t_{12}, T_2)/d\,p = 0$; that is, P_2 can be determined by solving the following equation:

$$\frac{dTP_2(p,t_{12},T_2)}{dp} = \frac{D(p)'}{T_2}\left\{(p-c+N)\left[t_{12} + \frac{ln[1+\delta(T_2-t_{12})]}{\delta}\right] - \frac{ht_{12}^2}{2} - NT_2\right\} + \frac{(D(p)+\mu)}{T_2}\left\{t_{12} + \frac{ln[1+\delta(T_2-t_{12})]}{\delta}\right\} \quad (4.16)$$

$$\frac{d^2TP_2(p,t_{12},T_2)}{dp^2} = \frac{D(p)''}{T_2}\left\{(p-c+M)\left[t_{12} + \frac{ln[1+\delta(T_2-t_{12})]}{\delta}\right] - \frac{ht_{12}^2}{2} - MT_2\right\} + \frac{2D(p)'}{T_2}\left\{t_{12} + \frac{ln[1+\delta(T_2-t_{12})]}{\delta}\right\}$$

Combining the previous Lemmas 4.2 and 4.4, the following result is obtained.

Theorem 4.5. *For any given p,*

a. *If* G(*p*) > 0, *the optimal length of time in which there is no inventory shortage is* t_{11} *and the optimal replenishment cycle length is* T_1,
b. *If* G(*p*) < 0, *the optimal length of time in which there is no inventory shortage is* t_{12} *and the optimal replenishment cycle length is* T_2,
c. *If* G(*p*) = 0, *the optimal length of time in which there is no inventory shortage is* t_d *and the optimal replenishment cycle length is* $t_d + h\, t_d / (\delta(p - c + N - h\, t_d))$.

Now, the following algorithm is established to obtain the optimal solution (p^*, t^*_1, T^*) of the problem.

5. Algorithm

Step 1. Start with $j = 0$ and the initial value of $P_j = c$.

Step 2. Calculate G(*pj*) = $(h\, t_d/\delta) - ((P_j - C + N)/\delta)\ \ln[(p_j - C + N)/(p_j - C + N - h\, t_d)] - (ht_d^2/2) + (K/(D(p_j)+\mu))$ for a *given* p_j,

i. if G(*pj*) > 0, determine the values $t_{11,j}$ and $T_{1,j}$ by solving (4.1). Then, put $(t_{11,j}, T_{1,j})$ into (4.11) and solve this equation to obtain the corresponding value $p_{1,j+1}$. Let $P_{j+1} = p_{1,j+1}$ and $(t_{1j}, T_j) = (t_{11,j}, T_{1,j})$, go to Step 3.
ii. If G(*pj*) < 0, determine the values $t_{l2,j}$ and $T_{2,j}$ by solving (4.12) and (4.13). Then, put $(t_{12,j}, T_{2,j})$ into (4.18) and solve this equation to obtain the corresponding value $P_{2,j+l}$. Let $P_{j+l} = P_{2,j+l}$ and $(t_{1j}, T_j) = (t_{12,j}, T_{2,j})$, go to Step 3.
iii. If G(*pj*) = 0, set $t_{1,j} = t_d$ and $T_j = t_d + (h\, t_d/\delta\ (p_j - C + N - h\, t_d))$, and then put $(t_{1,j}, T_j)$ into (4.11) or (4.18) to obtain the corresponding value $P_{1,j+l}(= P_{2,j+l})'$ Let $P_{j+l} = p_{1,j+l}$ or $P_{2,j+l}$ and $(t_{1j}'\ T_j) = (t_d,\ t_d + (h\, t_d/\delta\ (P_j - C + N - h\, t_d)))$, go to Step 3.

Step 3. If the difference between P_j and P_{j+l} is inevitable (for example, $|p_j - p_{j+l}| \le 0\ .005$), then set $p^* = P_j$ and $(t_1{*}\ T^*) = (t_{1j}, T_j)$. Thus (p^*, t^*_1, T^*) is the optimal solution. Otherwise, set $j = j + 1$ and go back to Step 2.

The above algorithm can be implemented with the help of a computer-oriented numerical technique for a given set of parameter values. Once (p^*, t_1^*, T^*) is obtained, Q* can be found from (3.11) or (3.13) and $TP^* = TP(p^*, t_1^*, T^*)$ from (3.12) or (3.14).

6. Numerical Example

In order to illustrate the solution procedure for this inventory system, the following example is presented.

h =$ 1 per unit/per unit time	S=$5/per unit/per unit time
O =$ 25/per unit	K = $250/per order
m =$23/per unit	C=$20/per unit
d (p)=200-4*p	Ɵ= 0.08
δ =0.1	ε~n(2, 1)
t_d= 0.08	

Under the given values of the parameters and according to the algorithm in the previous section, the computational results can be found after five iterations as follows, the optimal selling price p^* = $36.3812, the optimal length of time in which there is no inventory shortage t_1^* = 1.1360, and the optimal length of replenishment cycle T^* = 1.7123. Hence the optimal order quantity Q* = 98.3908 units, and the optimal total profit per unit time of the inventory system $TP(p^*, t^*_1, T^*)$ = $643.9107. The numerical results with distinct starting values of 35, 35.5, 36, 36.5, 37 and 37.5 were run. The numerical results reveal that TP is strictly concave in p, as shown in Fig. 2 As a result, we are sure that the local maximum obtained here from proposed algorithm is indeed the global maximum solution.

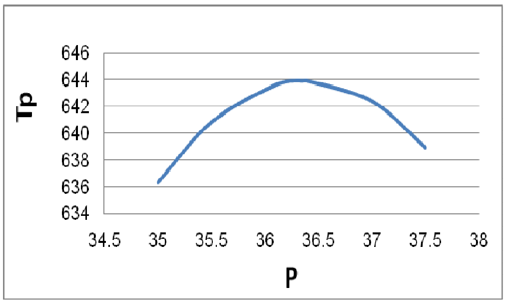


Figure 2. Graphical representation of TP (p| t_1^*, T^*)

Moreover, if t_d = 0, this model becomes the instantaneous deterioration case, and the optimal solutions can be found as follows: p^* = 36.4702, t_1^* = 1.1152, T^* = 1.7154, Q* = 98.1714, and TP^* = 633.6486. The results with instantaneous and non-instantaneous deterioration models for $t_d \in \{0, 0.08, 0.17 \text{ and } 0.24\}$ are shown in Table 1. From Table 1, it can be seen that there is an improvement in expected total profit from the non-instantaneously deteriorating model. Moreover, the longer the length of time where no deterioration occurs, the greater improvement in expected total profit will be. This implies that if the enterprises can extend the length of time in which no deterioration occurs by improving stock equipment, then the expected total profit per unit time will increase.

Table. 1. The results with instantaneous and non-instantaneous deteriorating models

t_d	p^*	t_1^*	T^*	Q*	TP^*
0	36.4702	1.1152	1.7154	98.1714	633.6486
0.08	36.3812	1.136	1.7123	98.3908	643.9107
0.17	36.2899	1.1621	1.7132	98.8445	654.8718
0.24	36.2166	1.1877	1.7179	99.2832	664.0866

This study now investigates the effects of changes in the values of the cost parameters *k*, C, h, s, o, and Ɵ on p^*, t^*_1, T^*, Q^* and $TP(p^*, t^*_1, T^*)$ according to above Example. The sensitivity analysis is performed by changing each value of the parameters by +50%, +25%, -25%, and -50%, taking one parameter at a time and keeping the remaining parameter values unchanged. The computational results are shown in Table 2.

On the basis of the results of Table 2, the following observations can be made.

When the values of parameters *k*, C, h, s, o and Ɵ increase, the optimal selling price p^* will increase. Moreover, p^* is weakly positively sensitive to changes in parameters *k*, h, s, o, and Ɵ, whereas p^* is highly positively sensitive to changes in parameter c. It is reasonable that the purchase cost has a strong and positive effect upon the optimal selling price.

Table. 1. Sensitive analysis with respect to the model parameters

parameter	% chnge	*p**	t_1*	T^*	*Q**	*TP**
K	-50	36.0131	0.8135	1.2026	70.532	730.3852
	-25	36.211	0.989	1.4781	85.7559	683.449
	25	36.534	1.2648	1.9203	109.3637	609.9398
	50	36.6745	1.3806	2.1096	119.1384	577.6357
C	-50	31.0898	1.2852	1.6836	134.7753	1351.613
	-25	33.7255	1.1865	1.6672	114.4997	968.019
	50	41.8562	1.1637	2.0301	70.0636	170.4891
	50	41.8562	1.1637	2.0301	70.0636	170.4891
h	-50	36.3304	1.2241	1.9133	109.703	653.6015
	-25	36.3848	1.209	1.7869	102.9492	649.8806
	25	36.4055	1.0943	1.6834	96.3494	638.2256
	50	36.4284	1.0559	1.6571	94.4951	632.8709
S	-50	36.2741	1.0707	1.8278	104.7497	660.6403
	-25	36.3352	1.1072	1.7613	101.0904	651.284
	25	36.4171	1.1593	1.6746	96.3117	637.9448
	50	36.4457	1.1785	1.6446	94.6573	633.0074
O	-50	36.3352	1.1072	1.7613	101.0904	651.284
	-25	36.3597	1.1224	1.7351	99.6343	647.3917
	25	36.4002	1.1482	1.6923	97.2877	640.7785
	50	36.4171	1.1593	1.6746	96.3117	637.9448
Θ	-50	36.2302	1.5479	2.0301	117.7414	686.8439
	-25	36.3156	1.3042	1.8388	106.1606	662.7933
	25	36.4336	1.0115	1.6221	92.779	628.5901
	50	36.4764	0.9148	1.5542	88.092	615.8625

- When the values of parameters *k*, C, h, s, o and Ɵ increase, the optimal selling price *p** will increase. Moreover, *p** is weakly positively sensitive to changes in parameters *k*, h, s, o, and Ɵ, whereas *p** is highly positively sensitive to changes in parameter c. It is reasonable that the purchase cost has a strong and positive effect upon the optimal selling price.
- In the case in which the values of parameters *k*, s, and o increase, the optimal length of time in which there is no inventory shortage t^*_1 will increase while it will decrease as the values of parameters h and Ɵ increase. From an economic viewpoint, this means that the retailer will avoid shortages when the order cost, shortage cost, and cost of lost sales are high.
- We showed that if the value of parameter *k* increase, the optimal length of the replenishment cycle *T** will increase, while it will decrease as the values of parameters h, s, o, and Ɵ increase. This implies that the higher the order cost the longer the length of the replenishment cycle, while the lower the holding cost, shortage cost, cost of lost sales, and deteriorating rate, the longer the length of the replenishment cycle.
- The optimal order quantity Q* will increase while the value of parameter *k* increase and it will decrease with an increase in the values of parameters c, h, s, o, and Ɵ. The corresponding managerial insight is that as the order cost increases, the order quantity increases. On the other hand, as the purchasing cost, holding cost, shortage cost, cost of lost sales, and deterioration rate increase, the order quantity decreases.
- When the values of parameters *k*, C, h, s, o, and Ɵ increase, the optimal total profit per unit time *T*P* will decrease. This implies that increases in costs and the deterioration rate have a negative effect upon the total profit per unit time.

7. Conclusions

The problem of determining the optimal replenishment policy for non-instantaneous deteriorating items with stochastic demand is considered in this study. In this system shortages are allowed and backlogging rate is variable which is defined as a function of waiting time for the next replenishment. There are two possible scenarios in this study: (1) the length of time in which there is no shortage is larger than or equal to the length of time in which the product exhibits no deterioration ($I_1 \geq t_d$) and (2) the length of time in which there is no shortage is shorter than or equal to the length of time in which the product exhibits no deterioration ($I_1 \leq t_d$). One numerical example is provided to illustrate the theoretical results under various situations and a sensitivity analysis of the optimal solution is performed with respect to major parameters. This paper contributes to existing methodology in several ways. Firstly, it addresses the problem of non-instantaneous deteriorating items under the circumstances in which the demand rate is stochastic and is price sensitive whereas the partial backlogging is allowed. Secondly, deterioration cost is considered that is not investigated in previous non-instantaneous models. This paper can be extended in several ways, for instance, we could extend model by considering the non- zero lead time. Also, we may consider the permissible delay in payment or promotions in the model.

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Appendix

A. Proof of Lemma 4.

Proof of part (a). It can be seen that F(x) is a strictly decreasing function in $x \in [t_d, t_1^b)$ and

$\lim_{x \to t_1^b} F(x) = -\infty$

Therefore, if $G(p) \equiv F(t_d) \geq 0$, then by using the Intermediate Value Theorem, there exists a unique value of t_1 (say t_{11})
such that $F(t_{11}) = 0$, that is, t_{11} is the unique solution of (4.4). Once the value t_{11} is found, then the value of T (denoted by T_1) can be found from (4.3) and given by

$$T_1 = t_{11} + \frac{M\left[e^{\theta(t_{11}-t_d)}-1\right]+ht_d}{\delta\{p-c+N-M\left[e^{\theta(t_{11}-t_d)}-1\right]-ht_d}$$

Proof of part (b). If $G(p) \equiv F(t_d) < 0$, then from F(x) is a strictly decreasing function of $x \in [t_d, t_1^b)$, which implies $F(x) < 0$ for all $x \in [t_d, t_1^b)$. Thus, a value $x \in [t_d, t_1^b)$ cannot be found such that $F(t_1) = 0$. This completes the proof.

B. Proof of lemma 4.2

Proof of part (a). For any given p, taking the second derivatives of $TP_1(t_1, T, p)$ with respect to t_1 and T and then finding the values of these function at point $(t_1, T) = (t_{11}, T_1)$ given

$$\frac{\partial^2 \boldsymbol{TP_1}(t_1,T,p)}{\partial t_1^2}\Big|_{(t_{11},T_1)} = \frac{D(p)+\mu}{T_1}\left\{\frac{-\delta(p-c+N)}{[1+\delta(T_1-t_{11})]^2} - Me^{\theta(t_{11}-t_d)}\right\} < 0$$

$$\frac{\partial^2 \boldsymbol{TP_1}(t_1,T,p)}{\partial T^2}\Big|_{(t_{11},T_1)} = \frac{D(p)+\mu}{T_1}\left\{\frac{-\delta(p-c+N)}{[1+\delta(T_1-t_{11})]^2}\right\} < 0$$

$$\frac{\partial^2 \boldsymbol{TP_1}(t_1,T,p)}{\partial T \partial t_1}\Big|_{(t_{11},T_1)} = \frac{D(p)+\mu}{T_1}\left\{\frac{\delta(p-c+N)}{[1+\delta(T_1-t_{11})]^2}\right\}$$

$$\frac{\partial^2 \boldsymbol{TP_1}(t_1,T,p)}{\partial t_1^2}\Big|_{(t_{11},T_1)} \times \frac{\partial^2 \boldsymbol{TP_1}(t_1,T,p)}{\partial T^2}\Big|_{(t_{11},T_1)} - \left[\frac{\partial^2 \boldsymbol{TP_1}(t_1,T,p)}{\partial T \partial t_1}\Big|_{(t_{11},T_1)}\right]^2 = \left(\frac{D(p)+\mu}{T_1}\right)^2\left\{\frac{\delta(p-c+N)Me^{\theta(t_{11}-t_d)}}{[1+\delta(T_1-t_{11})]^4}\right\} > 0$$

Because (t_{11}, T_1) is the unique solution of (4.1) if $G(p) \geq 0$, therefore, for any given p, (t_{11}, T_1) is the global maximum point of $TP_1(t_1, T, P)$.

Proof of part (b). For any given p, if $G(p) < 0$, then it is known that $F(x) < 0$, for all $x \in [t_d, t_1^b)$. Thus,

$$\frac{dTP_1(p,t_1,T)}{dT} = \frac{D(p)+\mu}{T^2}\Bigg\{-\left\{M\left[e^{\theta(t_1-t_d)}-1\right]+ht_d\right\}t_1 + \frac{M\left[e^{\theta(t_1-t_d)}-1\right]+ht_d}{\delta} - \frac{(p-c+M)}{\delta}\ln\left[\frac{p-c+N}{p-c+N-M\left[e^{\theta(t_1-t_d)}-1\right]-ht_d}\right] + \frac{M}{\theta} \times\left[e^{\theta(t_1-t_d)}-\theta(t_1-t_d)-1\right] + ht_dt_1 - \frac{ht_d^2}{2} + \frac{k}{D(p)+\mu}\Bigg\} = \frac{(D(p)+\mu)F(t_1)}{T^2} < 0$$

Which implies that $TP_1(t_1, T, p)$ is a strictly decreasing function of T. Thus, $TP_1(t_1, T, p)$ has a maximum value when T is minimum. On the other hand, form (4.3), it can be seen that T has a minimum value of
as $t_1 = t_d$. Therefore, $TP_1(t_1, T, p)$ has a maximum value at the point (t_{11}, T_1), where $t_{11} = t_d$ and

$$T_1 = t_d + \frac{ht_d}{\delta\{p-c+N-ht_d\}}.$$