The Optimal Inventory Policy with the Reusable Raw Material and Imperfect Items

Shou-Mei Su [1], Chia-Jung Chou [2], * Shy-Der Lin [3]

[1] *Department of Banking and Finance, Takming University of Science and Technology, Taipei City 11451, Taiwan, R.O.C.*

[2] *Department of Management Information System Takming University of Science and Technology, Taipei City 11451, Taiwan, R.O.C.*

[3] *Department of Applied Mathematics and Business Administration, Chung Yuan Christian University, Chung-Li City 32023, Taiwan, R.O.C.*

[1] meimei@takming.edu.tw

[2] raychou1999@gmail.com

* Corresponding author: [3] shyder@cycu.edu.tw

***Abstract*— This paper covers four topics regarding inventory models, namely reusable raw material, the EPQ model, imperfect-quality items, and the present value method. The relevant cost value used in traditional EPQ (economic production quantity) models does not include the stockholding cost of raw material, which makes such models unsuitable for investigating production. And we are not sure that all of the products are perfect items. Because people in the world are attempting to reduce the impact of environmental impairment and increasing market competition, all products are manufactured from 100% reusable raw material and are screened during the manufacturing process. By taking the fixed proportion of imperfect-quality items and the time value into account and applying the present value method to analyze optimal inventory policies, this study creates a modified EPQ inventory model that is close to real life we meet. Furthermore, this model aims to promote the reputation of a company and ascertain its costs accurately.**

***Keywords*—** reusable, raw material, EPQ, imperfect-quality items, present value

1. Introduction

The reuse of materials and products has long been considered in the subject of academic study. By using repaired and newly made products, Richter [1] created a model in order to find the most cost-effective fixed and variable collection time intervals. Later, Richter and Dobos [2] extended the model of Richter [3] by incorporating integer setup numbers. By applying Pontryagin's Maximum Principle, Kleber et al. [4] further determined the optimal production, remanufacturing, and disposal policy for a cost model. Koh et al. [5] then created a joint EOQ and EPQ model in which a fixed proportion of used products was collected from customers and recovered for reuse. Konstantaras and Papachristos [6] revised Koh et al.'s [5] paper and used a different analysis technique to obtain closed form expressions for the optimal setup number in both the recovery and the ordering processes. Karakayal et al. [7] then characterized the optimal acquisition price of used products and the selling price as well as the recovery quantities of the reusable components.

Salameh and Jaber [8] extended the traditional EPQ/EOQ model by accounting for imperfect-quality items when using the EPQ/EOQ formula. Their paper also considered the issue that poor quality items were being sold as a single batch at a lower price compared with good quality items at the end of the 100% screening process. In Schwatter's [9] research, a fixed percentage of imperfect-quality items and a fixed or variable screening cost were added into the model. Chen et al. [10] investigated the learning effect of the unit production time on optimal lot size for imperfect production systems with allowable shortages. Chen and Kang [11] developed integrated vendor–buyer models that considered a permissible delay in payment and imperfect-quality in order to determine the optimal solutions of a buyer's order quality and the frequency of each vendor's production.

The EOQ model was first proposed by Harris [12] and later the EPQ model was developed by E. W.

International Journal of Supply Chain Management
IJSCM, ISSN: 2050-7399 (Online), 2051-3771 (Print)

Taft [13]. The EPQ model is a well-known and commonly used in inventory control technique. E.W. Taft [13] did not consider the stock-holding cost for the raw materials. In today's factory, producers are required to prepare raw materials, parts, … etc, that will be used in the production in order to finish the operation schedule. These prepared raw materials, parts, … etc. will cause extra stock-holding cost, therefore, we add the stock-holding cost in the EPQ model.

Up to now Salameh & El-Kassar [14] had the paper to establish an EPQ model taking the stock-holding cost of raw material into consideration and found the optimal lot size. El-Kassaret et al [15] studied an EPQ model for imperfect quality raw material.

The research of Trippi [16], Kim et al [17], Moon and Yun [18], and Chung and Lin [19] all mentioned the time value of money. Kim et al [17] presented a method for evaluating investment in inventory. Teng [18] used the discounting cash-flow approach to establish the models, and obtained the optimal ordering policies to the problem. Moon and Yun [18] justified the optimality of those solutions derived from the first-order conditions in Kim et al [17]. Then Chung and Lin [21] refuted the concavity of net present value for infinite planning horizon and conclusions expressed or implied by Kim et al [17]. Later, Chung and Lin [21] derived the bounds for the optimal cycle following the optimality of solutions. Using upper and lower bounds, they developed an algorithm for computing the optimal cycle time.

In this study, we create a model based on Richter's [1] idea of reusable items and Salameh and Jaber's [8] notion of imperfect-quality items. We also follow the research of Chung and Lin [19] on the time value of money and adopt their algorithm to ascertain the optimal cycle time (order size). These numerical examples were included in the algorithm in order to explore the different cycle times. We have referenced these previous studies to ensure that the models presented in this paper are appropriate and practical.

2. Definition and assumptions

The mathematical models developed in this study are based on the following notations and assumptions.

Notations:

PVC(T): the present value of the cash flow for the first inventory horizon

$PVC_{\infty}(T)$: the present value of the cash flow for the infinite planning horizon

TRC(T): the total relevant cost per unit time

Q:the order size,

S: the cost of placing an order,

P: the production rate,

D: the demand rate,

C: the purchasing cost per unit,

α : the percentage of defective items in finished production,

β : the reusable rate,

x: the screening rate,

d: the unit screening cost,

T: the cycle length,

r: the discount rate,

h_1 : the stock holding cost of raw materials per item per year,

h_2 : the stock holding cost of finished products per item per year,

b: the selling price of unit imperfect-quality items,

t_1 : the screening time.

Assumptions:

(1) Production rate is greater than demand rate.
(2) Production rate and demand rate are known and constant.
(3) Percentage of defective items and screening rate are known and constant.
(4) Shortage is not allowed.
(5) A single item is considered.
(6) Time horizon is infinite.
(7) The stock holding cost of raw materials and products are set separately.
(8) $0 \le \alpha < 1$.
(9) $0 < \beta \le 1$.
(10) $\frac{D}{(1-\alpha)} \le x \le P$ (The range of screening time t_1 is $\frac{DT}{(1-\alpha)P} \le t_1 \le T$ where $t_1 = \frac{DT}{(1-\alpha)x}$).
(11) The imperfect-quality items are sold as a

single batch at the end of screening process and the money is received at the end of cycle time.

3. The models

Two models are illustrated in the following (See Figure 1). First, we use the annual total relevant cost, TRC(T). Second, we use the present value of total relevant cost for infinite planning horizon, PVC_{∞} (T). We will find the optimal cycle time separately for each model, and then use the numerical examples to find the optimal value of cycle time.

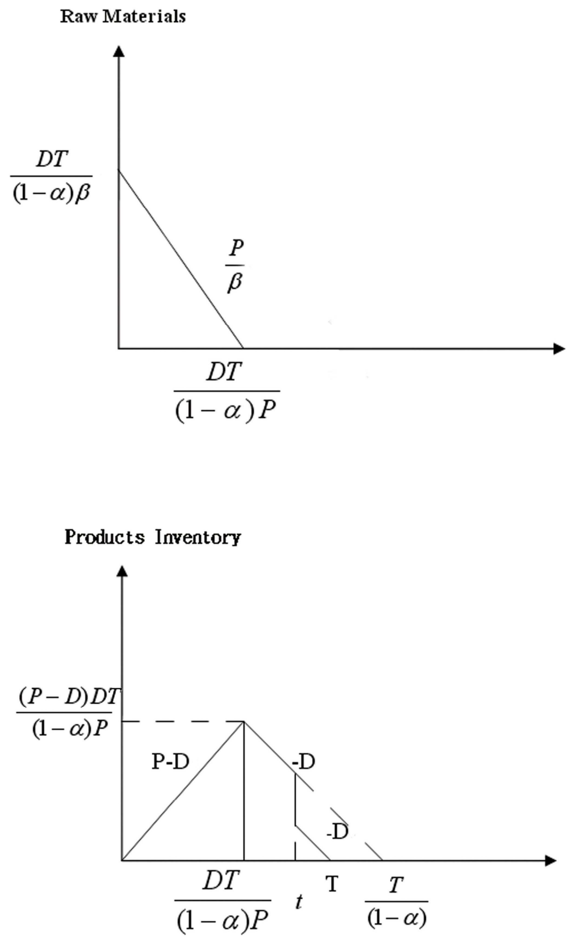

Figure 1 The EPQ model containing imperfect-quality items

Model 1: **Using the annual total relevant cost to find the optimal solution**

The annual total relevant cost TRC(T) consists of the following elements:

The ordering cost per unit time = $\frac{S}{T}$.

The purchasing cost per order per unit time $= \frac{cD}{(1-\alpha)\beta}$.

The screening cost per unit time = $\frac{dD}{(1-\alpha)}$.

The stock holding cost of raw materials per unit time

$$= \frac{h_1 D^2 T}{2(1-\alpha)^2 \beta P}.$$

The stock holding cost of products per unit time =

$$h_2\left[\frac{(P-D)D^2T}{2(1-\alpha)^2P^2}+\frac{D^2T}{2(1-\alpha)^2}(2-\frac{D}{x}-\frac{D}{P})(\frac{1}{x}-\frac{1}{P})+\frac{DT}{2}(1-\frac{D}{(1-\alpha)x})^2\right]$$

The money earned from selling the imperfect-quality items per unit time = $\frac{\alpha bD}{(1-\alpha)}$.

The total relevant cost per unit time can be expressed as follows:

TRC(T) = the ordering cost per unit time + the purchasing cost per order per unit time + the screening cost per unit time +the stock holding cost of raw materials per unit time-the money earned from selling the imperfect-quality items per unit time + the stock holding cost of products per unit time.

$$\text{TRC(T)} = \frac{S}{T} + \frac{cD}{(1-\alpha)\beta} + \frac{dD}{(1-\alpha)} + \frac{h_1 D^2 T}{2(1-\alpha)^2 \beta P} - \frac{\alpha bD}{(1-\alpha)} + h_2[\frac{DT}{2}(1-\frac{D}{(1-\alpha)x})^2] + h_2\left[\frac{(P-D)D^2T}{2(1-\alpha)^2P^2}+\frac{D^2T}{2(1-\alpha)^2}(2-\frac{D}{x}-\frac{D}{P})(\frac{1}{x}-\frac{1}{P})\right] \quad (1)$$

And the first and second derivatives of TRC(T) are

$$\frac{dTRC(T)}{dT} = -\frac{S}{T^2} + \frac{h_1 D^2}{2(1-\alpha)^2\beta P} + h_2\left[\frac{(P-D)D^2}{2(1-\alpha)^2P^2}+\frac{D^2}{2(1-\alpha)^2}(2-\frac{D}{x}-\frac{D}{P})(\frac{1}{x}-\frac{1}{P})\right] + h_2[\frac{D}{2}(1-\frac{D}{(1-\alpha)x})^2] \quad (2)$$

and $\frac{d^2TRC(T)}{dT^2} = \frac{2S}{T^3} > 0$ for all T>0.

Set $\frac{dTRC(T)}{dT} = 0$ and set W =

$$\frac{h_1D^2}{2(1-\alpha)^2\beta P} + h_2[\frac{D}{2}(1-\frac{D}{(1-\alpha)x})^2] + h_2\left[\frac{(P-D)D^2}{2(1-\alpha)^2P^2} + \frac{D^2}{2(1-\alpha)^2}(2-\frac{D}{x}-\frac{D}{P})(\frac{1}{x}-\frac{1}{P})\right] \quad (3)$$

If $\frac{dTRC(T)}{dT} = 0$ then we have the following result:

$$-\frac{S}{T^2} + W = 0. \quad (4)$$

The unique solution of equation (4) is $T^{**} = \sqrt{\frac{S}{W}}$.

Since $\frac{d^2TRC(T)}{dT^2} > 0$ for all T>0, the TRC(T) has a global minimum on $(0, \infty)$ at the time of $T = T^{**}$.

Model 2: **Using the present value of total relevant cost for infinite planning horizon $PVC_\infty(T)$ to find the optimal solution.**

The present value of total relevant cost at the first cycle time PVC(T) consists of the following elements:

The ordering cost = S.

The purchasing cost = $\frac{cDT}{(1-\alpha)\beta}$.

The screening cost = $\frac{dDT}{(1-\alpha)}$.

The stock holding cost of raw materials

$$= h_1\int_0^{\frac{DT}{(1-\alpha)P}}[\frac{DT}{(1-\alpha)\beta} - \frac{P}{\beta}t]e^{-rt}dt.$$

The stock holding cost of products=

$$h_2\{\int_0^{\frac{DT}{(1-\alpha)P}}(P-D)te^{-rt}dt + \int_{\frac{DT}{(1-\alpha)P}}^{\frac{DT}{(1-\alpha)x}}D[\frac{T}{(1-\alpha)} - t]e^{-rt}dt + \int_{\frac{DT}{(1-\alpha)x}}^{T}D(T-t)e^{-rt}dt\}.$$

The money earned from selling the imperfect-quality items = $\frac{\alpha bDT}{(1-\alpha)}e^{-rT}$.

The present value of total relevant cost at the first cycle time PVC(T) is as followed:

$$PVC(T) = S + \frac{cDT}{(1-\alpha)\beta} + \frac{dDT}{(1-\alpha)} + h_1\int_0^{\frac{DT}{(1-\alpha)P}}[\frac{DT}{(1-\alpha)\beta} - \frac{P}{\beta}t]e^{-rt}dt +$$

$$h_2\{\int_0^{\frac{DT}{(1-\alpha)P}}(P-D)te^{-rt}dt + \int_{\frac{DT}{(1-\alpha)P}}^{\frac{DT}{(1-\alpha)x}}D[\frac{T}{(1-\alpha)} - t]e^{-rt}dt + \int_{\frac{DT}{(1-\alpha)x}}^{T}D(T-t)e^{-rt}dt\}.$$

$$= S + \frac{cDT}{(1-\alpha)\beta} + \frac{dDT}{(1-\alpha)} + \frac{h_1DT}{(1-\alpha)\beta r} - \frac{h_1P}{\beta r^2} + \frac{h_1P}{\beta\cdot r^2}e^{-\frac{rDT}{(1-\alpha)P}} + \frac{h_2(P-D)}{r^2} - \frac{h_2P}{r^2}e^{-\frac{rDT}{(1-\alpha)P}} - \frac{h_2DT}{(1-\alpha)r}e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2DT}{r}e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2D}{r^2}e^{-rT} - \frac{\alpha bDT}{(1-\alpha)}e^{-rT} \quad (5)$$

Now we want to find the present value of total relevant cost for infinite planning horizon $PVC_\infty(T)$

where $PVC_\infty(T) = \sum_{i=0}^{\infty} PVC(T)e^{-irT} = \frac{PVC(T)}{1-e^{-rT}}$

$$= (1-e^{-rT})^{-1}[S + \frac{cDT}{(1-\alpha)\beta} + \frac{dDT}{(1-\alpha)} + \frac{h_1DT}{(1-\alpha)\beta r} - \frac{h_1P}{\beta r^2} + \frac{h_1P}{\beta r^2}e^{-\frac{rDT}{(1-\alpha)P}} + \frac{h_2(P-D)}{r^2} - \frac{h_2P}{r^2}e^{-\frac{rDT}{(1-\alpha)P}} - \frac{h_2DT}{(1-\alpha)r}e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2DT}{r}e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2D}{r^2}e^{-rT} - \frac{\alpha bDT}{(1-\alpha)}e^{-rT}]. \quad (6)$$

The first derivative of $PVC_\infty(T)$ is

$$\frac{dPVC_\infty(T)}{dT} = (1-e^{-rT})^{-2}\ f(T) \qquad (7)$$

where $f(T) = (1-e^{-rT})[\ \frac{cD}{(1-\alpha)\beta} + \frac{dD}{(1-\alpha)} + \frac{h_1D}{(1-\alpha)\beta r} - \frac{h_1D}{(1-\alpha)\beta r}\ e^{-\frac{rDT}{(1-\alpha)P}} + \frac{h_2D}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)P}}$

$- \frac{h_2D}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2D^2T}{(1-\alpha)^2x}\ e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2D}{r} e^{-\frac{rDT}{(1-\alpha)x}} - \frac{h_2D^2T}{(1-\alpha)x} e^{-\frac{rDT}{(1-\alpha)x}} - \frac{h_2D}{r}\ e^{-rT} - \frac{\alpha bD}{(1-\alpha)} e^{-rT} + \frac{\alpha brDT}{(1-\alpha)} e^{-rT}$] - re^{-rT} [S + $\frac{cDT}{(1-\alpha)\beta} + \frac{dDT}{(1-\alpha)} + \frac{h_1DT}{(1-\alpha)\beta r} - \frac{h_1P}{\beta r^2} + \frac{h_1P}{\beta r^2}\ e^{-\frac{rDT}{(1-\alpha)P}} + \frac{h_2(P-D)}{r^2}$

$- \frac{h_2P}{r^2}\ e^{-\frac{rDT}{(1-\alpha)P}} - \frac{h_2DT}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2DT}{r}\ e^{-\frac{rDT}{(1-\alpha)x}}$

$+ \frac{h_2D}{r^2}\ e^{-rT} - \frac{\alpha bDT}{(1-\alpha)} e^{-rT}$]. (8)

Then $\frac{dPVC_\infty(T)}{dT}$ =0 if and only if $f(T)$=0.

Let $\frac{dPVC_\infty(T)}{dT} = \frac{f(T)}{(1-e^{-rT})^2} = \frac{e^{-rT} g(T)}{(1-e^{-rT})^2}$, (9)

which immediately implies that

$\frac{dPVC_\infty(T)}{dT}$, $f(T)$ and $g(T)$ have the same sign (+ or -) with time T.

$g(T) = e^{rT}\ f(T)$

$= (1-e^{-rT})$ [$\frac{cD}{(1-\alpha)\beta}\ e^{rT} + \frac{dD}{(1-\alpha)}\ e^{rT} + \frac{h_1D}{(1-\alpha)\beta r}\ e^{rT} - \frac{h_1D}{(1-\alpha)\beta r}\ e^{-\frac{rDT}{(1-\alpha)P}+rT} + \frac{h_2D}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)P}+rT} - \frac{h_2D}{r}$

$- \frac{h_2D}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)x}+rT} + \frac{h_2D^2T}{(1-\alpha)^2x}\ e^{-\frac{rDT}{(1-\alpha)x}+rT} - \frac{\alpha bD}{(1-\alpha)}$

$+ \frac{h_2D}{r}\ e^{-\frac{rDT}{(1-\alpha)x}+rT} - \frac{h_2D^2T}{(1-\alpha)x}\ e^{-\frac{rDT}{(1-\alpha)x}+rT} + \frac{\alpha brDT}{(1-\alpha)}$]

- r [S + $\frac{cDT}{(1-\alpha)\beta} + \frac{dDT}{(1-\alpha)} + \frac{h_1DT}{(1-\alpha)\beta r} - \frac{h_1P}{\beta r^2}$

$+ \frac{h_1P}{\beta r^2}\ e^{-\frac{rDT}{(1-\alpha)P}} + \frac{h_2(P-D)}{r^2} - \frac{h_2P}{r^2}\ e^{-\frac{rDT}{(1-\alpha)P}} - \frac{h_2DT}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2DT}{r}\ e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2D}{r^2}\ e^{-rT} - \frac{\alpha bDT}{(1-\alpha)} e^{-rT}$]. (10)

Moreover, we have $\lim_{T\to 0^+} g(T)$ = -rS<0 (11)

and $\lim_{T\to\infty} g(T) = \infty$. (12)

Theorem1: $g'(T) > 0$

Proof:

$g'(T)$ =re^{-rT} [$\frac{cD}{(1-\alpha)\beta}\ e^{rT} + \frac{dD}{(1-\alpha)}\ e^{rT} + \frac{h_1D}{(1-\alpha)\beta r}\ e^{rT} - \frac{h_1D}{(1-\alpha)\beta r}\ e^{-\frac{rDT}{(1-\alpha)P}+rT} + \frac{h_2D}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)P}+rT}$

$- \frac{h_2D}{(1-\alpha)r}\ e^{-\frac{rDT}{(1-\alpha)x}+rT} + \frac{h_2D^2T}{(1-\alpha)^2x}\ e^{-\frac{rDT}{(1-\alpha)x}+rT}$

$$+ \frac{h_2D}{r} e^{-\frac{rDT}{(1-\alpha)x}+rT} - \frac{h_2D^2T}{(1-\alpha)x} e^{-\frac{rDT}{(1-\alpha)x}+rT} - \frac{h_2D}{r}$$

$$- \frac{\alpha bD}{(1-\alpha)} + \frac{\alpha brDT}{(1-\alpha)}] + (1-e^{-rT}) [\frac{cDr}{(1-\alpha)\beta} e^{rT}$$

$$+ \frac{dDr}{(1-\alpha)} e^{rT} - \frac{h_1D(1-\frac{D}{(1-\alpha)P})}{(1-\alpha)\beta} e^{-\frac{rDT}{(1-\alpha)P}+rT}$$

$$+ \frac{h_1D}{(1-\alpha)\beta} e^{rT} + \frac{h_2D(1-\frac{D}{(1-\alpha)P})}{(1-\alpha)} e^{-\frac{rDT}{(1-\alpha)P}+rT}$$

$$- \frac{h_2D(1-\frac{D}{(1-\alpha)x})}{(1-\alpha)} e^{-\frac{rDT}{(1-\alpha)x}+rT} + \frac{h_2D^2}{(1-\alpha)^2x} \cdot e^{-\frac{rDT}{(1-\alpha)x}+rT}$$

$$+ \frac{h_2rD^2T(1-\frac{D}{(1-\alpha)x})}{(1-\alpha)^2x} e^{-\frac{rDT}{(1-\alpha)x}+rT} + \frac{\alpha brD}{(1-\alpha)}$$

$$+ h_2D(1-\frac{D}{(1-\alpha)x}) e^{-\frac{rDT}{(1-\alpha)x}+rT} - \frac{h_2D^2}{(1-\alpha)x} e^{-\frac{rDT}{(1-\alpha)x}+rT} -$$

$$\frac{h_2rD^2T(1-\frac{D}{(1-\alpha)x})}{(1-\alpha)x} e^{-\frac{rDT}{(1-\alpha)x}+rT}]-r [\frac{cD}{(1-\alpha)\beta} +$$

$$\frac{dD}{(1-\alpha)} + \frac{h_1D}{(1-\alpha)\beta r} - \frac{h_1D}{(1-\alpha)\beta r} e^{-\frac{rDT}{(1-\alpha)P}} +$$

$$\frac{h_2D}{(1-\alpha)r} e^{-\frac{rDT}{(1-\alpha)P}} - \frac{h_2D}{(1-\alpha)r} e^{-\frac{rDT}{(1-\alpha)x}} +$$

$$\frac{h_2D^2T}{(1-\alpha)^2x} e^{-\frac{rDT}{(1-\alpha)x}} + \frac{h_2D}{r} e^{-\frac{rDT}{(1-\alpha)x}} -$$

$$\frac{h_2D^2T}{(1-\alpha)x} e^{-\frac{rDT}{(1-\alpha)x}} - \frac{h_2D}{r} e^{-rT} - \frac{\alpha bD}{(1-\alpha)} e^{-rT} + \frac{\alpha brDT}{(1-\alpha)} e^{-rT}]$$

$$= (1-e^{-rT})$$

$$[\frac{cDr}{(1-\alpha)\beta} e^{rT} + \frac{dDr}{(1-\alpha)} e^{rT} + \frac{h_1D}{(1-\alpha)\beta} e^{rT} -$$

$$\frac{h_1D(1-\frac{D}{(1-\alpha)P})}{(1-\alpha)\beta} e^{-\frac{rDT}{(1-\alpha)P}+rT} +$$

$$\frac{h_2D^2}{(1-\alpha)^2x} e^{-\frac{rDT}{(1-\alpha)x}+rT}$$

$$+ \frac{h_2D(1-\frac{D}{(1-\alpha)P})}{(1-\alpha)} e^{-\frac{rDT}{(1-\alpha)P}+rT} -$$

$$\frac{h_2D(1-\frac{D}{(1-\alpha)x})}{(1-\alpha)} e^{-\frac{rDT}{(1-\alpha)x}+rT} +$$

$$\frac{h_2rD^2T(1-\frac{D}{(1-\alpha)x})}{(1-\alpha)^2x} e^{-\frac{rDT}{(1-\alpha)x}+rT}$$

$$+ h_2D(1-\frac{D}{(1-\alpha)x}) e^{-\frac{rDT}{(1-\alpha)x}+rT} -$$

$$\frac{h_2D^2}{(1-\alpha)x} e^{-\frac{rDT}{(1-\alpha)x}+rT} -$$

$$\frac{h_2rD^2T(1-\frac{D}{(1-\alpha)x})}{(1-\alpha)x} e^{-\frac{rDT}{(1-\alpha)x}+rT} + \frac{\alpha brD}{(1-\alpha)}]$$

$$= (1-e^{-rT}) \{ \frac{cDr}{(1-\alpha)\beta} e^{rT} + \frac{dDr}{(1-\alpha)} e^{rT}$$

$$+ \frac{h_1D}{(1-\alpha)\beta} e^{rT} [1-(1-\frac{D}{(1-\alpha)P}) e^{--\frac{rDT}{(1-\alpha)P}}]$$

$$+ \frac{h_2De^{(1-\frac{D}{(1-\alpha)P})rT}}{(1-\alpha)} [1 - \frac{D}{(1-\alpha)P} - (1 - \frac{D}{(1-\alpha)x}) e^{(\frac{rDT}{(!-\alpha)P}-\frac{rDT}{(1-\alpha)x})}] + \frac{h_2D^2}{(1-\alpha)x} (\frac{1}{1-\alpha} - 1) e^{-\frac{rDT}{(1-\alpha)x}+rT}$$

$$+ \frac{\alpha brD}{(1-\alpha)} + h_2D(1-\frac{D}{(1-\alpha)x}) e^{-\frac{rDT}{(1-\alpha)x}+rT}$$

$$+ \frac{h_2rD^2T}{(1-\alpha)x} (1-\frac{D}{(1-\alpha)x})(\frac{1}{1-\alpha} - 1) e^{-\frac{rDT}{(1-\alpha)x}+rT} \}>0 \tag{13}$$

Thus we have proved that $g'(T) > 0$. (14)

So $g(T)$ is a strictly increasing function.

From equations (11), (12) and (14), we conclude that $g(T)=0$ has a unique solution T^* which satisfying

$$g(T)\begin{cases}<0 \ \ if \ \ 0<T<T^* \\ =0 \ \ if \ \ \ T=T^* \\ >0 \ \ if \ \ \ T^*<T\end{cases} \qquad (15)$$

and since $\frac{dPVC_\infty(T)}{dT} = \frac{e^{-rT}g(T)}{(1-e^{-rT})^2}$, so it implies that there is a unique solution T^* which satisfying

$$\frac{dPVC_\infty(T)}{dT}\begin{cases}<0 \ \ if \ \ 0<T<T^* \\ =0 \ \ if \ \ \ T=T^* \\ >0 \ \ if \ \ \ T^*<T\end{cases} \qquad (16)$$

It is not easy to solve T^* exactly out. We try to find an upper bound and a lower bound of T^* first. And then use Intermediate Value Theorem and the algorithm of bisection method to find the optimal cycle time. We will show the procedure of finding a lower bound T_L^* and an upper bound T_U^*.

To find a lower bound T_L^*

$$g(T) = e^{rT}\ [\ \frac{cD}{(1-\alpha)\beta}(1-e^{-rT}) + \frac{dD}{(1-\alpha)}(1-e^{-rT})$$

$$+\frac{h_1 D}{(1-\alpha)\beta r}(1-e^{-rT}) - \frac{h_1 D}{(1-\alpha)\beta r}e^{-\frac{rDT}{(1-\alpha)P}}(1-e^{-rT})$$

$$+\frac{h_2 D}{(1-\alpha)r}e^{-\frac{rDT}{(1-\alpha)P}}(1-e^{-rT}) - \frac{h_2 D}{(1-\alpha)r}e^{-\frac{rDT}{(1-\alpha)x}}(1-e^{-rT})$$

$$+\frac{h_2 D^2 T}{(1-\alpha)^2 x}e^{-\frac{rDT}{(1-\alpha)x}}(1-e^{-rT})$$

$$+\frac{h_2 D}{r}e^{-\frac{rDT}{(1-\alpha)x}}(1-e^{-rT})$$

$$-\frac{h_2 D^2 T}{(1-\alpha)x}e^{-\frac{rDT}{(1-\alpha)x}}(1-e^{-rT}) - \frac{h_2 D}{r}e^{-rT}(1-e^{-rT}) - \frac{\alpha bD}{(1-\alpha)}e^{-rT}(1-e^{-rT})$$

$$+\frac{\alpha brDT}{(1-\alpha)}e^{-rT}(1-e^{-rT})\ ]$$

$$-\ \mathrm{r\,S}\ e^{-rT} - \frac{crDT}{(1-\alpha)\beta}e^{-rT} - \frac{drDT}{(1-\alpha)}e^{-rT}$$

$$-\frac{h_1 DT}{(1-\alpha)\beta}e^{-rT} + \frac{h_1 P}{\beta r}e^{-rT} - \frac{h_1 P}{\beta r}e^{-\frac{rDT}{(1-\alpha)P}-rT}$$

$$-\frac{h_2(P-D)}{r^2}e^{-rT} + \frac{h_2 P}{r}e^{-\frac{rDT}{(1-\alpha)P}-rT}$$

$$+\frac{h_2 DT}{(1-\alpha)}e^{-\frac{rDT}{(1-\alpha)x}-rT} - h_2 DT\ e^{-\frac{rDT}{(1-\alpha)x}-rT}$$

$$-\frac{h_2 D}{r}e^{-2rT} - \frac{\alpha bDT}{(1-\alpha)}e^{-2rT}\ ]$$

$$< e^{rT}\ \{[\ \frac{h_1 rD}{2(1-\alpha)\beta} + \frac{h_2 rD^3}{2(1-\alpha)^3P^2} + \frac{cr^2D}{(1-\alpha)\beta}$$

$$+\frac{dr^2D}{(1-\alpha)} + \frac{h_1 rD}{(1-\alpha)\beta} + \frac{h_1 rP}{2\beta} + \frac{2\alpha bD}{(1-\alpha)}\ ]\,T^2 +$$

$$[\ \frac{h_1 D^2}{(1-\alpha)^2\beta P} + \frac{h_2 D^2}{(1-\alpha)x} + \frac{2h_2 D}{(1-\alpha)} + \frac{2h_2 D^2}{(1-\alpha)^2 x}$$

$$+h_2 P + r^2 S + 3\,h_2 D + \frac{2\alpha brD}{(1-\alpha)} - \frac{\alpha bD}{(1-\alpha)}\ ]\,T - \mathrm{rS}\ \}. \qquad (17)$$

Setting

$$A_1 = \frac{h_1 rD}{2(1-\alpha)\beta} + \frac{h_2 rD^3}{2(1-\alpha)^3P^2} + \frac{cr^2D}{(1-\alpha)\beta}$$

$$+\frac{dr^2D}{(1-\alpha)} + \frac{h_1 rD}{(1-\alpha)\beta} + \frac{h_1 rP}{2\beta} + \frac{2\alpha bD}{(1-\alpha)}\ (>0), \qquad (18)$$

$$B_1 = \frac{h_1 D^2}{(1-\alpha)^2\beta P} + \frac{h_2 D^2}{(1-\alpha)x} + \frac{2h_2 D}{(1-\alpha)} + \frac{2h_2 D^2}{(1-\alpha)^2 x} + h_2 P +$$

$$r^2 s + 3\,h_2 D + \frac{2\alpha brD}{(1-\alpha)} - \frac{\alpha bD}{(1-\alpha)}, \qquad (19)$$

$$C_1 = -\ \mathrm{rS}\ (<0), \qquad (20)$$

thus, we conclude that

$$g(T) < e^{rT}\ (A_1 T^2 + B_1 \mathrm{T} + C_1). \qquad (21)$$

The positive solution of $e^{rT}\,(A_1 T^2 + B_1 T + C_1) = 0$

is T_L^*

where
$$T_L^* = \frac{-B_1+\sqrt{B_1^2-4A_1C_1}}{2A_1} \qquad (22)$$

To find an upper bound T_U^*

$$g(T) = e^{rT-\frac{rDT}{(1-\alpha)P}}\Big[\frac{cD}{(1-\alpha)\beta}\, e^{\frac{rDT}{(1-\alpha)P}} - \frac{cD}{(1-\alpha)\beta}\, e^{\frac{rDT}{(1-\alpha)P}-rT}$$

$$+\frac{dD}{(1-\alpha)}\, e^{\frac{rDT}{(1-\alpha)P}} - \frac{dD}{(1-\alpha)}\, e^{\frac{rDT}{(1-\alpha)P}-rT} + \frac{h_1D}{(1-\alpha)\beta r}\, e^{\frac{rDT}{(1-\alpha)P}}$$

$$-\frac{h_1D}{(1-\alpha)\beta r}\, e^{\frac{rDT}{(1-\alpha)P}-rT} - \frac{h_1D}{(1-\alpha)\beta r} + \frac{h_1D}{(1-\alpha)\beta r}\, e^{-rT}$$

$$+\frac{h_2D}{(1-\alpha)r} - \frac{h_2D}{(1-\alpha)r}\, e^{-rT} - \frac{h_2D}{(1-\alpha)r}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}} +$$

$$\frac{h_2D}{(1-\alpha)r}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}-rT} + \frac{h_2D^2T}{(1-\alpha)^2x}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}}$$

$$-\frac{h_2D^2T}{(1-\alpha)^2x}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}-rT} + \frac{h_2D}{r}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}}$$

$$-\frac{h_2D}{r}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}-rT} - \frac{h_2D^2T}{(1-\alpha)x}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}}$$

$$+\frac{h_2D^2T}{(1-\alpha)x}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}-rT} - \frac{h_2D}{r}\, e^{\frac{rDT}{(1-\alpha)P}-rT}$$

$$+\frac{h_2D}{r}\, e^{\frac{rDT}{(1-\alpha)P}-2rT}(1-e^{-rT}) -$$

$$\frac{\alpha bD}{(1-\alpha)}\, e^{\frac{rDT}{P}-rT}(1-e^{-rT}) + \frac{\alpha brDT}{(1-\alpha)}\, e^{\frac{rDT}{P}-rT}(1-e^{-rT})$$

$$-\,r\,S\,e^{\frac{rDT}{(1-\alpha)P}-rT} - \frac{crDT}{(1-\alpha)\beta}\, e^{\frac{rDT}{(1-\alpha)P}-rT} - \frac{drDT}{(1-\alpha)}\, e^{\frac{rDT}{(1-\alpha)P}-rT}$$

$$-\frac{h_1DT}{(1-\alpha)\beta}\, e^{\frac{rDT}{(1-\alpha)P}-rT} + \frac{h_1P}{\beta r}\, e^{\frac{rDT}{(1-\alpha)P}-rT} - \frac{h_1P}{\beta r}\, e^{-rT}$$

$$-\frac{h_2(P-D)}{r^2}\, e^{\frac{rDT}{(1-\alpha)P}-rT} + \frac{h_2P}{r}\, e^{-rT}$$

$$+\frac{h_2DT}{(1-\alpha)}\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}-rT} - h_2DT\, e^{(\frac{1}{P}-\frac{1}{x})\frac{rDT}{(1-\alpha)}-rT}$$

$$-\frac{h_2D}{r}\, e^{\frac{rDT}{(1-\alpha)P}-2rT} - \frac{\alpha bDT}{(1-\alpha)}\, e^{\frac{rDT}{(1-\alpha)P}-2rT}\Big]$$

$$> e^{rT-\frac{rDT}{(1-\alpha)P}}\Big\{\frac{rD^3}{2(1-\alpha)^3P^2}\Big(\frac{cr}{\beta}+dr+\frac{h_1}{\beta}\Big)T^2 + \Big[\Big(\frac{D^2}{(1-\alpha)^2P}-\frac{D}{(1-\alpha)}\Big)\Big(\frac{cr}{\beta}+dr+\frac{h_1}{\beta}\Big) -$$

$$\frac{h_2D^2}{(1-\alpha)x}\Big(\frac{1}{1-\alpha}\Big) - h_2D\Big]T -$$

$$\Big(\frac{h_1D}{(1-\alpha)\beta r}+\frac{h_2D}{(1-\alpha)r}+\frac{h_2D}{r}+rS+\frac{h_1P}{\beta r}$$

$$+\frac{h_2P}{r}-\frac{\alpha bD}{(1-\alpha)}\Big)\Big\} \qquad (23)$$

Set
$$A_2 = \frac{rD^3}{2(1-\alpha)^3P^2}\Big(\frac{cr}{\beta}+dr+\frac{h_1}{\beta}\Big)\ (>0), \qquad (24)$$

$$B_2 = \Big(\frac{D^2}{(1-\alpha)^2P}-\frac{D}{(1-\alpha)}\Big)\Big(\frac{cr}{\beta}+dr+\frac{h_1}{\beta}\Big)$$

$$-\frac{h_2D^2}{(1-\alpha)x}\Big(\frac{1}{1-\alpha}\Big) - h_2D, \qquad (25)$$

$$C_2 = -\Big(\frac{h_1D}{(1-\alpha)\beta r}+\frac{h_2D}{(1-\alpha)r}+\frac{h_2D}{r}+rS+\frac{h_1P}{\beta r}$$

$$+\frac{h_2P}{r}+\frac{\alpha bD}{(1-\alpha)}\Big)\ (<0). \qquad (26)$$

So we get that

$$g(T) > e^{rT - \frac{rDT}{(1-\alpha)P}} \left(A_2\, T^2 + B_2\, T + C_2 \right). \qquad (27)$$

The positive solution of $e^{rT - \frac{rDT}{(1-\alpha)P}} (A_2\, T^2 + B_2\, T + C_2) = 0$ is T_U^*

where $T_U^* = \dfrac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}$. (28)

Theorem2: $T_L^* < T^* < T_U^*$

Proof:

Since $g(T)$ is a strictly increasing function, we have

(1). By equation (21)

$$g(T_L^*) < e^{rT_L^*} (A_1\, T_L^{*2} + B_1\, T_L^* + C_1) = 0 = g(T^*) \qquad (29)$$

So $g(T_L^*) < g(T^*)$ (30)

and $T_L^* < T^*$. (31)

(2). By Equation (27) $g(T_U^*) > e^{rT_U^* - \frac{rDT_U^*}{(1-\alpha)P}} (A_2\, T_U^{*2} + B_2\, T_U^* + C_2) = 0 = g(T^*)$ (32)

So $g(T_U^*) > g(T^*)$ (33)

and $T^* < T_U^*$. (34)

From the equations (31) and (34), we have $T_L^* < T^* < T_U^*$.

Now we can compute the exact optimal cycle length T^* by using the logic of the following algorithm. Our method is similar to the one of Chung and Lin (see [21]).

Step 1: Let $\varepsilon > 0$

Step 2: Set $T_L^* = \dfrac{-B_1 + \sqrt{B_1^2 - 4A_1C_1}}{2A_1}$ and $T_U^* = \dfrac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}$

where $A_1 = \dfrac{h_1 rD}{2(1-\alpha)\beta} + \dfrac{h_2 rD^3}{2(1-\alpha)^3 P^2} + \dfrac{cr^2 D}{(1-\alpha)\beta} + \dfrac{dr^2 D}{(1-\alpha)} + \dfrac{h_1 rD}{(1-\alpha)\beta} + \dfrac{h_1 rP}{2\beta} + \dfrac{2\alpha bD}{(1-\alpha)}$,

$B_1 = \dfrac{h_1 D^2}{(1-\alpha)^2 \beta P} + \dfrac{h_2 D^2}{(1-\alpha)x} + \dfrac{2h_2 D}{(1-\alpha)} + \dfrac{2h_2 D^2}{(1-\alpha)^2 x} + h_2 P + r^2 s + 3\,h_2 D + \dfrac{2\alpha brD}{(1-\alpha)} - \dfrac{\alpha bD}{(1-\alpha)}$,

$C_1 = -\,rS$.

And $A_2 = \dfrac{rD^3}{2(1-\alpha)^3 P^2} \left(\dfrac{cr}{\beta} + dr + \dfrac{h_1}{\beta} \right) > 0$,

$B_2 = \left(\dfrac{D^2}{(1-\alpha)^2 P} - \dfrac{D}{(1-\alpha)} \right) \left(\dfrac{cr}{\beta} + dr + \dfrac{h_1}{\beta} \right) - \dfrac{h_2 D^2}{(1-\alpha)x} \left(\dfrac{1}{1-\alpha} \right) - h_2 D$,

$C_2 = -\left(\dfrac{h_1 D}{(1-\alpha)\beta r} + \dfrac{h_2 D}{(1-\alpha)r} + \dfrac{h_2 D}{r} + rS + \dfrac{h_1 P}{\beta r} + \dfrac{h_2 P}{r} + \dfrac{\alpha bD}{(1-\alpha)} \right)$.

Step 3: Set $T_{opt} = \dfrac{T_L^* + T_U^*}{2}$

Step 4: If $\left| K(T_{opt}) \right| < \varepsilon$, go to Step 6.

Otherwise go to Step5.

Step 5: If $f(T_{opt}) > 0$, then we set $T_U^* = T_{opt}$,

And if $f(T_{opt}) < 0$, then we set $T_L^* = T_{opt}$.

Then go to Step3

Step 6: $T^* = T_{opt}$,

where T_{opt} is the optimal cycle length.

4. Numerical examples

Example1: If we set the numbers as S=1000, P=2000, D=1500, c=10, α =0.1, β =0.8, x=1800, d=0.5, r=0.05, h_1 =2, h_2 =1.5, b=5 and ε =0.00001. Then we have T^{**} =0.69827 and T^{*} = 0.61539

Example2: If we set the numbers as S=1000, P=2000, D=1500, c=10, α =0.1, β =0.8, x=1800, d=0.5, r=0.03, h_1 =2, h_2 =1.5, b=5 and ε =0.00001. Then we have T^{**} =0.69827 and T^{*} = 0.64495

Example3: If we set the numbers as S=1000, P=2000, D=1500, c=10, α =0.1, β =0.8, x=1800, d=0.5, r=0.1, h_1 =2, h_2 =1.5, b=5 and ε =0.00001. Then we have T^{**} =0.6213885 and T^{*} = 0.55598

Example4: If we set the numbers as S=1000, P=2000, D=1500, c=10, α =0.2, β =0.8, x=1800, d=0.5, r=0.05, h_1 =2, h_2 =1.5, b=5 and ε =0.00001. Then we have T^{**} =0.621389 and T^{*} = 0.55298

Example5: If we set the numbers as S=1000, P=2000, D=1500, c=10, α =0.1, β =0.5, x=1800, d=0.5, r=0.05, h_1 =2, h_2 =1.5, b=5 and ε =0.00001. Then we have T^{**} =0.56864 and T^{*} = 0.49976

Example6: If we set the numbers as S=1000, P=2000, D=1500, c=10, α =0.1, β =1, x=1800,d=0.5, r=0.05, h_1 =2, h_2 =1.5, b=5 and ε =0.00001. Then we have T^{**} =0.76613 and T^{*} = 0.67644

Example7: If we set the numbers as S=1000, P=2000, D=1500, c=10, α =0, β =0.8, x=1800, d=0.5, r=0.05, h_1 =2, h_2 =1.5, b=5 and ε =0.00001. Then we have T^{**} =0.84327 and T^{*} = 0.74040

5. Discussion

Since the range of β (the reusable rate) is $0 < \beta \le 1$. Example 6 (the reusable rate equals 1) shows that the model still works well when using new raw materials instead of using reusable materials. And since the range of α is $0 \le \alpha < 1$. Example 7 (the percentage of defective items in finished production) shows the result of having perfect products. It still works if we assume that the model has perfect items.

6. Conclusions

Since the interest rate is lower than it was previously, adding the inflation variable is suggested in future research. The rate of imperfect-quality items could also be replaced by a variable value. We are unsure whether the stockholding costs of raw materials and of products are the same; if they were the same, the model would be simpler. The models are established in order to extend the practicality and usability of the traditional EPQ model. This study thus provides an approach to increase the perfect quality of productions in order to promote the reputation of a company and reduce the risk of inventory cost.

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