Bullwhip Effect Variance Ratio Approximations for Aggregated Retail Orders in Supply Chains

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***Abstract*— Aggregated retailers' total order to the supplier is one of the factors that contribute to the Bullwhip effect in two-echelon supply chains. Though the estimation of the variance of the total of orders is hard to be computed, this paper contribution is to provide two approximations to the ratio of the aggregated retailer orders variance at the supplier side with respect to the market demand variance. Using these approximations few managerial insights can be considered in real practice based on retailers' (S, s) inventory policies and a supplier fixed periodic review inventory policy. In addition an optimal review period is derived that minimizes the variance ratio in the case of low market demand rate.**

Keywords*—Supply chain management, bullwhip effect, variance estimation, Aggregated Orders, Bullwhip effect variance ratio*

1. Introduction

The Bullwhip Effect in Supply Chain management is a widely recognized phenomenon in supply chain management where many researchers attempted various approaches in order to study its circumstances and understand its consequences. One of the first of works that was conducted on this phenomenon is found in [1] where a Markov chain model for the aggregated demand of retailers was developed assuming (S, s) inventory policies used by retailers. The variance of the total order quantity of individual retailer during a fixed time period was explicitly derived in this paper, as well as the variance of the aggregated orders as received by the supplier. This same paper also proved the statistical independence of retailers' orders.

The authors of [2] considered an optimization approach to derive the optimal production decisions under demand uncertainty. The name "Bullwhip Effect" was first introduced [3] to call the variance distortion of the supplier production. The authors of this work based their findings on empirical data and specified few factors that lead to such phenomenon. A mathematical model was suggested in [4] to help stabilize this phenomenon. On its part, [5] proposed some approximation tools to analyze the effect of changes in various parameters of the (S, s) inventory policies that would lead to the variability in aggregated orders upstream the supply chain.
Most recently, simulation models were developed in [6] to test various inventory policies on reducing the bullwhip effect in multi-echelon supply chains. A portfolio theory was used in [7] to adjust the order quantities based on their order variance, and it was argued in [8] that certain replenishment policies can stimulate the Bullwhip effect while other may actually reduce it. The latter suggested few rules to select the appropriate policy that would mitigate the bullwhip effect to occur. Several scenarios concerning lead times between warehousing and retailers were examined in [9] to investigate the impact of third party warehousing on three-echelon supply chain order variance. Later on, it was showed in [10] that aggregation of retailer orders over long periods can hide the size of the Bullwhip phenomenon and emphasized the need to choose appropriate times to identify its existence.

This paper establishes two approximations for the bullwhip effect variance ratios of the aggregated retailers' orders, where the first is based on the renewal theory of the market demand process and the second approximation uses existing results that are based on Markov chain models both for two-echelon supply chains. The exact derivation of the variance of aggregated retailers' orders proved to be very hard. The paper considers simple Poisson process of the

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market demand where items are ordered in units and assumes as well that the market is partitioned among a number of independent retailers.

The two approximations that are considered could be realized in two different situations of the market demand process. The first assumes low demand rate and uses the results of [1], while the other is uses a second-order approximation of the variance of a renewal process at limiting conditions of high demand rates as derived in [11] and more elaborated in [12] is applied. This paper provides direct results of these two approximations and suggests managerial insights and suggests policies that may exist in practice.

2. The Market Renewal Process Model

A simple two-echelon supply chain that consists of a single supplier and few retailers is assumed for simplicity where the market demand process occurs according to a homogeneous Poisson process with constant demand rate equal to λ.

Consider m retailers who share the market according to a probability distribution p_i for $i = 1,2, \dots, m$. The fact that the market demand follows a homogeneous Poisson distribution leads to the result that each retailer will entertain his shared demand in the market independently from the other retailers and with fixed rate equal to $p_i\lambda$

Also assume that the (S, s) inventory policy is common for all retailers with different order quantities Q_i, $i = 1,2, \dots, m$, set independently by each retailer. Clearly all order quantities Q_i's are constant due to the assumption of the (S, s) policies. All lead times are assumed zero so that the focus will remain only on the impact of the order quantities on the variance of the total of all retailers' orders as received by the supplier.

Since the aim of this paper is to estimate the variance of the total of all orders made by the supplier's customers, i.e the retailers, within a fixed period equals to R, which can be considered to be the review period in a periodic review policy used by the supplier.

Under these assumptions, let $\{Y_t, t > 0\}$ be the market demand process which is a Poisson process. The mean demand during period R is therefore λR.

Similarly each retailer market demand will follow a Poisson process, which during the supplier's review period R will have a mean demand equals to $p_i\lambda R$ This is true because we assumed the Poisson market demand process and consequently, the retailers' shares in the market form a partition.

On the other hand, the fact that the shares of the suppliers in the market demand follow Poisson distribution yields to the fact that the times between two subsequent sales at a retailer are independently and exponentially distributed. As each retailer makes a replenishment order to the supplier only after Q_i units are sold to the market, this means that the times between his replenishment orders follow a random variable, say T_i, which is the sum of a number of Q_i independent exponentially distributed random times. Therefore, T_i are distributed according to a Gamma density function with parameters Q_i and $p_i\lambda R$, and this is the same for every retailer i.

Furthermore, while all the times between retailer's orders are independent, then the ordering process by each retailer i is subject to a renewal process, say $\{N_i(t), t > 0\}$, that counts the number of orders he makes during the supplier's review period R. Let the total quantity of such retailer's orders during R be defined by $X_i(R) = Q_i N_i(R)$, then consequently, the aggregated orders to the supplier from all retailers is the total $X(R) = \sum_{i=1}^{m} X_i(R)$. The goal now is to estimate the mean and variance of the aggregated total of all retailers' orders $X(R)$

By the renewal theorem, the mean of $N_i(t)$ is

$$E[N_i(R)] = \frac{p_i\lambda R}{Q_i} \ . \qquad (1)$$

Using (1) to estimate the mean of $X(R)$ to get the mean of the whole market demand, it follows that

$$\begin{aligned} E[X(R)] &= \sum_{i=1}^{m} E[X_i(R)] \\ &= \sum_{i=1}^{m} Q_i \, E[N_i(R)] \\ &= \sum_{i=1}^{m} Q_i \frac{p_i\lambda R}{Q_i} \end{aligned}$$

$$= \lambda R \sum_{i=1}^{m} p_i$$

$$= \lambda R, \qquad (2)$$

which is an obvious result. But on the other hand,

$$VAR[X(R)] = \sum_{i=1}^{m} VAR[X_i(R)]$$

$$= \sum_{i=1}^{m} Q_i^2 \, VAR[N_i(R)]. \qquad (3)$$

Since $\{N_i(t), t > 0\}$ is a renewal process, the computation of its variance is essentially hard even for the simple Poisson process.

This paper aims to develop two approximations to the variance of $X_i(R)$: The first approximation considers a Markov chain model for the retailers inventories developed by [1] and uses it for low market demand rates. The second approximation makes use of a second-order approximation for the variance of a renewal process as derived by [11] and investigated further by [12], and can be applied for large supplier's review period R or high market demand λ.

3. The Markov Chain model Approximation Scheme for Low Market Demand Rate

In [1], the variance of the sales for retailer i was derived by a Markov chain approach under the assumption that $\sum_{y=0}^{Q_i-1} f_i(y) = 1$, where $f_i(\cdot)$ is the probability density function of the market sales at retailer i. This can be used as an approximation when the market demand rate λ is low.

The lemma below proves that the tail probabilities of the Poisson distribution for low demand rate λ can be approximately considered to be zero

As a consequence, similar to the finding in [1], the number of orders that are set by retailer i during the supplier's review period R will be either 0 or 1. Hence, $E[N_i(R)] = E[N_i^2(R)]$ and this in turns leads to

$$Var[N_i(R)] = E[N_i^2(R)] - (E[N_i(R)])^2$$

$$= \frac{p_i \lambda R}{Q_i} - \left(\frac{p_i \lambda R}{Q_i}\right)^2$$

$$= \left(\frac{p_i \lambda R}{Q_i}\right)\left[1 - \left(\frac{p_i \lambda R}{Q_i}\right)\right] \qquad (4)$$

Therefore,

$$Var[X_i(R)] = Q_i^2 Var[N_i(R)]$$

$$= p_i Q_i \lambda R \left[1 - \left(\frac{p_i \lambda R}{Q_i}\right)\right] \qquad (5)$$

Since all retailers' sales are mutually independent as they make an independent partition of the total Poisson market demand process, then

$$Var[X(R)] = \lambda R \sum_{i=1}^{m} p_i Q_i \left[1 - \left(\frac{p_i \lambda R}{Q_i}\right)\right] \quad (6)$$

Knowing that the market demand variance is λR, then the Bullwhip effect variance ratio, say γ_1, is

$$\gamma_1 = \frac{Var[X(R)]}{Var[Y(R)]}$$

$$= \sum_{i=1}^{m} p_i Q_i \left[1 - \left(\frac{p_i \lambda R}{Q_i}\right)\right] \qquad (7)$$

Recall however, that this result requires the condition in [1] where $\sum_{y=0}^{Q_i-1} f_i(y) = 1$, which is not exactly fulfilled for the Poisson demand probability distribution but can however be approximated as it will be shown in the following Lemma.

Based on the fact that the above ratio $\gamma_1$cannot be negative, then for it to be valid, it requires $Q_i \geq p_i \lambda R$. The following Lemma shows that this condition approximately meet the requirements of the results in [1] simply because in this case $f_i(y) \approx 0$ for $y \geq Q_i$.

Lemma: *Given large enough positive integers α and M for any $y \geq M$*

$$M \geq \alpha \Rightarrow f_i(y) = e^{-\alpha} \frac{\alpha^y}{y!} \approx 0 \qquad (8)$$

Proof: Take $\alpha \geq 1$ an integer, for a Poisson distribution $f_i(y)$ we have

$$\frac{f_i(\alpha+1)}{f_i(\alpha)} = e^{-1}\left(\frac{\alpha+1}{\alpha}\right)^{\alpha} = e^{-1}\left(1 + \frac{1}{\alpha}\right)^{\alpha} \qquad (9)$$

Knowing that $lim_{\alpha \to \infty}\left(1 + \frac{1}{\alpha}\right)^{\alpha} = e$, it is enough to prove that $\left(1 + \frac{1}{\alpha}\right)^{\alpha}$ is monotonically increasing to deduce that $\left(1 + \frac{1}{\alpha}\right)^{\alpha} < e$ for every finite α.

For this purpose, the first derivative for the function $g(x) = \left(1 + \frac{1}{x}\right)^{x}$ is

$g^{'}(x) =$

$$\left(1+\frac{1}{x}\right)^{x-1}\left[\left(1+\frac{1}{x}\right)Ln\left(1+\frac{1}{x}\right)+\frac{1}{x}\right] \qquad (10)$$

It is then clear that $g^{'}(x) > 0$ for $x \geq 0$. Thus the function $g(x)$ is increasing for $x \geq 0$. This is especially true for integer numbers. ■

The above result also means that $f_i(\alpha+1) < f_i(\alpha)$ for every finite integer α, and since each $f_i(\alpha) \geq 0$, being probability terms, then consequently $\lim_{\alpha\to\infty} f_i(\alpha) = 0$ and therefore $f_i(\alpha) = e^{-\alpha}\frac{\alpha^\alpha}{\alpha!} \approx 0$ for large α. Furthermore, for even a bigger integer M where $M > \alpha$ it is becomes clear that $f_i(y) = e^{-\alpha}\frac{\alpha^y}{y!} \approx 0$ when $y \geq M \geq \alpha$.

The result in the above lemma is therefore applicable when every retailer orders a constant quantity $Q_i \geq p_i\lambda R$, where the probability terms for orders bigger than or equal to Q_i will be almost zero, and thus the above approximation becomes valid.

4. The Second-Order Renewal Theory Approximation

A second-order linear approximation to the variance of the counting process for a renewal process was worked out in [11] as well as [12] to be:

$$Var[N_i(t)] \approx at + b \qquad (11)$$

as $t \to \infty$, where

$$a = \mu_1^{-3}(\mu_2 - \mu_1^{-2}) = \mu_1^{-3}\sigma^2, \qquad (12)$$

and

$$b = \frac{5}{4}\mu_1^{-4}\mu_2^2 - \frac{2}{3}\mu_1^{-3}\mu_3 + 2\mu_1^{-2}\mu_2 - 3\mu_1^{-3}\mu_2\mu_1 + \frac{1}{2}\mu_1^{-2}\mu_2 \qquad (13)$$

Here, μ_r is the r^{th} moment and σ^2 is the variance of the renewal function for which the counting process $N_i(t)$ is defined.

By computing these moments and the variance for the Gamma process whose scale parameter is the retailer order quantity Q_i, and shape parameter is equal to the Poisson market demand rate $p_i\lambda$, we obtain

$$\mu_r = \frac{Q_i(Q_i+1)(Q_i+2)\cdots(Q_i+r)}{(p_i\lambda)^r}. \qquad (14)$$

When substituted in the approximation of the variance of the number of a retailer orders, we get

$$a = \frac{p_i\lambda}{Q_i^2} \text{ and } b = \frac{1}{12}\left(1-\frac{1}{Q_i^2}\right). \qquad (15)$$

Therefore,

$$Var[N_i(t)] \approx \frac{p_i\lambda}{Q_i^2}\, t + \frac{1}{12}\left(1-\frac{1}{Q_i^2}\right). \qquad (16)$$

So, the variance of the order quantity during a review period R for the supplier will be

$$Var[X_i(R)] = Q_i^2\, Var[N_i(R)] \approx p_i\lambda R + \frac{1}{12}(Q_i^2 - 1). \qquad (17)$$

The variance of the total orders received by the supplier from all retailers during the supplier's review period is

$$Var[X(R)] = \sum_{i=1}^m Var[X_i(R)] \approx \sum_{i=1}^m p_i\lambda R + \frac{1}{12}(Q_i^2 - 1), \qquad (18)$$

that is,

$$Var[X(R)] \approx \lambda R + \sum_{i=1}^m \frac{(Q_i^2-1)}{12}. \qquad (19)$$

Therefore, the second approximation for the Bullwhip effect is derived by dividing the supplier order variance as obtained above, by the Poisson market demand variance λR, thus

$$\gamma_2 \approx 1 + \frac{1}{\lambda R}\sum_{i=1}^m \frac{1}{12}(Q_i^2 - 1)\,. \qquad (20)$$

This result is useful when the review cycle R is large. It is also useful when the market demand rate λ is high where the assumption of [1] is no longer valid. In either case, multiple retailer orders to the supplier is possible during the supplier review period.

5. Managerial Implications

The following discuss the practical implications for special cases regarding the bullwhip effect ratios obtained in (7) and (20).

5.1. Case 1: Market Dominating Retailer

When one retailer dominates a given market over other competitors, that is that market sales partition of the market is reduced to a single retailer while all other shares are practically zero. This is reflected in

$$\gamma_1 = Q\left(1 - \frac{\lambda R}{Q}\right) = Q - \lambda R, \qquad (21)$$

where Q is the order quantity for this retailer. Obviously, as it was assumed that $Q > \lambda R$, then γ_1 is obviously larger than one and gets bigger as Q increases. It is therefore recommended in this case that the supplier should consider the acquisition of this retailer or attempts to interfere in its operations. Policies such as Vendor Managed Inventory can also be used in order to have better control on the order variance amplification due to the Bullwhip effect.

5.2. Case 2: Equal market shares by retailers

Another extreme situation concerning retailers competition is realized when all retailers' shares in the market are equal. In this case, $p_i = \frac{1}{m}$, where m is the number of retailers. Here the bullwhip effect ratio becomes

$$\gamma_1 = \frac{1}{m}\left(\sum_{i=1}^{m} Q_i - \lambda R\right). \qquad (22)$$

Define $\bar{Q} = \frac{\sum_{i=1}^{m} Q_i}{m}$ to be the average of all retailers order quantities, then $\gamma_1 = \bar{Q} - \frac{\lambda R}{m}$. It is clear then that when the number of retailers m increases, the bullwhip effect variance ratio also increases unless the supplier management work on lowering the retailers order quantities in order to reduce $\bar{Q}$ at the same time.

5.3. Case 3: Large market demand rate

When the market demand rate λ increases, the first approximation of the Bullwhip effect ratio will no longer be appropriate since the required condition in [1] will no longer holds. Therefore, the second second-order renewal approximation of the bullwhip effect variance ratio, i.e. γ_2, is hereby considered. It is clear that this ratio asymptotically approaches to 1.

It is assumed in this case that the review period R is constant and that the retailers' order quantities Q_i, for $i = 1, \cdots, m$ are finite.

It can also be argued that when the market demand rate increases, it will lead to an increasing number of retailers in the market and thus, the order quantities by these retailers will remain finite. Hence, it is recommended that the supplier should increase the number of retailers in the market in case that the market demand increases.

5.4. Long Supplier Review Periods

Similar result to Case 3 above is also true when the supplier review period is long enough to have multiple single retailer orders during one review cycle. However, this policy is not preferred as it increases the warehousing costs to the supplier in order to meet adequate service levels to the retailers. The following section actually supports the argument that longer review periods are not adequate in common practice.

6. Optimum Review Period for Low Market Demand in the Case of EOQ Retailer Order Quantity

In an extension to the above analysis, if one can assume as mostly common that each retailer order is proportional to the square root of the market sales mean rate, that is $Q_i = C_i\sqrt{p_i \lambda R}$, where C_i is a constant, then the derivative of γ_1 with respect to R is

$$\frac{\partial \gamma_1}{\partial R} = \sum_{i=1}^{m} \frac{C_i p_i}{2\sqrt{p_i \lambda R}} - \lambda R \sum_{i=1}^{m} p_i^2. \qquad (23)$$

Setting (23) to zero to get the optimal review period as

$$R^* = \frac{\sum_{i=1}^{m} \frac{C_i}{2\lambda}\sqrt{\frac{p_i}{\lambda}}}{\sum_{i=1}^{m} p_i^2}. \qquad (24)$$

As it shows, R^* is inversely proportional to the market demand rate λ. That means that products with higher market demand rate require smaller inventory review period. This will ultimately lead to a continuous review policy, which will be better than the periodic review policy in such case.

7. Conclusion

In this paper two approximations were derived for the ratio of the aggregated independent retail orders variance to the market demand subject to the Bullwhip Effect. The first approximation considers low market demand rate situation while the second uses renewal theory approximation for the case where the market demand is high and/or the supplier's review period is long. Some useful managerial insights were discussed in each situation. Future work needs to take place to extend these results to correlated retailer orders to the supplier. Bounds on the variance ratios may also be the aim of future research. Lastly, an optimum review period for the supplier is derived in the case of low market demand rate.

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